Searching for the QCD critical point using Lee-Yang edge singularities

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The lattice approach: No fermions

Want expected value of operator O.

- In lattice QCD, expectation values given by

$$\langle O\rangle = \frac{1}{\mathcal{Z}}\int \mathcal{D}U\, e^{-S(U)}\, O(U), \qquad \mathcal{Z} \equiv \int \mathcal{D}U\, e^{-S(U)}$$

- Markov Chain Monte Carlo basic idea:
  - Each configuration generated depending on last one only
  - Accept new configuration with probability $\min\{1, e^{-\Delta S}\}$
  - Create a time series of measurements O_n of O
- The estimator for $\langle O\rangle$ on the lattice is

$$\bar{O} = \frac{1}{N_{\text{conf}}}\sum_{n=1}^{N_{\text{conf}}} O_n$$

A little more detail when there's fermions

$$
\begin{aligned}
\langle O \rangle &\sim \int \mathrm{d}\bar{\psi}\,\mathrm{d}\psi\, e^{\bar{\psi}D\psi} \quad \mathrm{d}U\, e^{-S(U)}\, O(U) \\
&= \int \det D \qquad\qquad \mathrm{d}U\, e^{-S(U)}\, O(U)
\end{aligned}
$$

Complication:

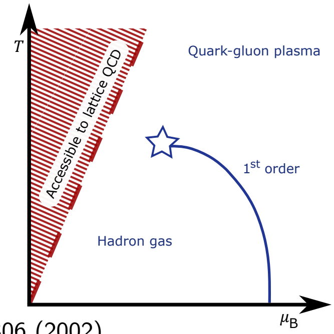
- $\det D \in \mathbb{R}$ when $\mu = 0$
- But if $\mu \neq 0$, it is in general complex...

The infamous problem

Trick: μ_B pure imaginary avoids sign problem; can analytically continue to $\mu_B \in \mathbb{R}$[1,2].

Trick: Expand pressure P/T^4 in μ_B/T[3,4].

The latter is getting a bit too pricey. Popularity of resummation schemes[5,6,7,8].



[1] P. de Forcrand and O. Philipsen, Nuclear Physics B, 642.1-2, 290–306 (2002).
[2] M. D'Elia and M.-P. Lombardo, Phys. Rev. D, 67.1, 014505 (2003).
[3] C. R. Allton et al., Phys. Rev. D, 66.7, 074507 (2002).
[4] R. V. Gavai and S. Gupta, Phys. Rev. D, 68.3, 034506 (2003).
[5] S. Borsányi et al., Phys. Rev. Lett. 126.23, 232001 (2021).
[6] D. Bollweg et al., Phys. Rev. D, 105.7, 074511 (2022).
[7] S. Mitra, P. Hegde, and C. Schmidt, Phys. Rev. D, 106.3, 034504 (2022).
[8] S. Mondal, S. Mukherjee, and P. Hegde, Phys. Rev. Lett. 128.2, 022001 (2022).

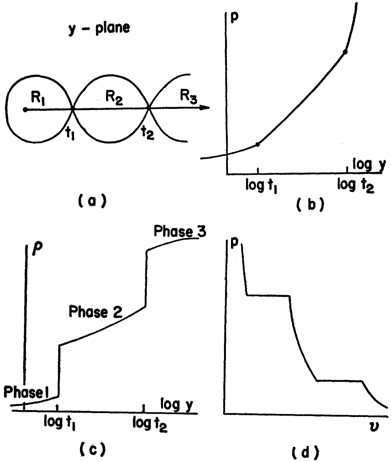
Lee-Yang theorem

Works where $\log \mathcal{Z}_{\rm QCD}$ is free of singularities.

Lee-Yang theorem[9]: Zeroes of the partition function that approach the real axis as $V \to \infty$ correspond to phase transitions.

Intuition: Indications of non-analyticities in P

- may hint at phase transitions
- or singularities in $\mathbb{C}$
- constrain validity of Taylor series



[9] C. N. Yang and T. D. Lee, Phys. Rev. 87.3, 404–409 (1952).

Lee-Yang edges and extended analyticity

Ising: Generically have branch cuts on imaginary axis. (Pinch real axis at T_c.)

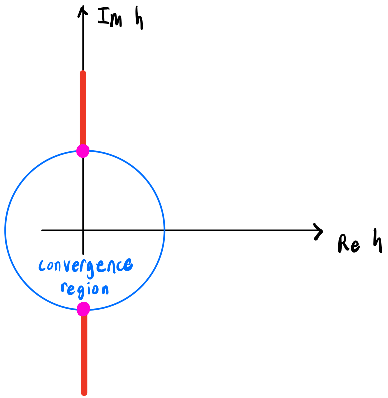
Lee-Yang edge (LYE): The singularities closest to real axis.

Extended analyticity conjecture[10]: LYE is the nearest singularity to the origin.

LYE position fixed at

$$z_c = |z_c| e^{\pm i\pi/2\beta\delta}$$

with $z \equiv th^{-1/\beta\delta}$ and critical exponents β, δ.



[10] P Fonseca and A Zamolodchikov, J. Stat. Phys. 110, 527–590 (2003).

Padé approximants

Want detailed information about singularities ⇒ **rational functions**,

$$R^m_n(x) \equiv \frac{\sum_{i=0}^m a_i x^i}{1+\sum_{j=1}^n b_j x^j}.$$

Singularities captured or mimicked by zeros in denominator.

Let f have a formal Taylor series

$$f(x) = \sum_{k=0}^{\infty} c_k x^k.$$

Padé approximant of order $[m, n]$: R^m_n with coefficients so that it equals the Taylor series up to order $m+n$. Gives relationship between coefficients a_i, b_j, c_k.

Padé approximants

Things to think about with Padé:

- Theorem: Unique when it exists
- Theorem: $[m, n]$ converges to f exactly as $m \to \infty$ when f has pole of order n
- Other properties deduced from numerical experiments
- Limited by number of known Taylor coefficients
- Only have up to 8^{th} order[11,12] for $\log \mathcal{Z}_{\text{QCD}}$; difficultly far greater for higher orders[13]

[11] S. Borsanyi et al., J. High Energ. Phys. 2018.10, 205 (2018).
[12] D. Bollweg et al., Phys. Rev. D, 108.1, 014510 (2023).
[13] Computational requirements of HotQCD EoS exceed 2000 GPU-years and 2.4 PB.

Multi-point Padé approximants

Padé approximants you get by demanding[14]

$$R_n^m(x) = f^{m+n}(x) \equiv \sum_{i=0}^{m+n} c_k x^k.$$

Say we know Taylor series up to some order s. The Multi-point Padé is the R_n^m satisfying

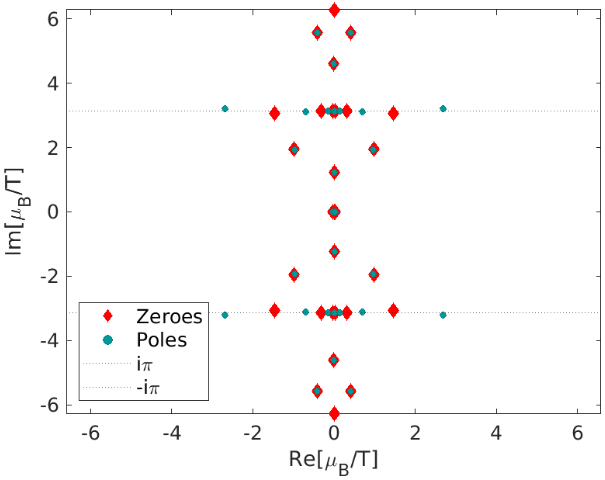
$$\left.\frac{\mathrm{d}^l R_n^m}{\mathrm{d}x^l}\right|_{x_i} = \left.\frac{\mathrm{d}^l f}{\mathrm{d}x^l}\right|_{x_i}$$

for N points x_i, $0 \leq l < s-1$. Some pros/cons:

- Need fewer Taylor coefficients!
- Less seems to be known about them...

[14] One expects corresponding relationships among derivatives of R and f.

Extracting a LYE[15]



[15] P. Dimopoulos et al., Phys. Rev. D, 105.3, 034513 (2022).

The strategy

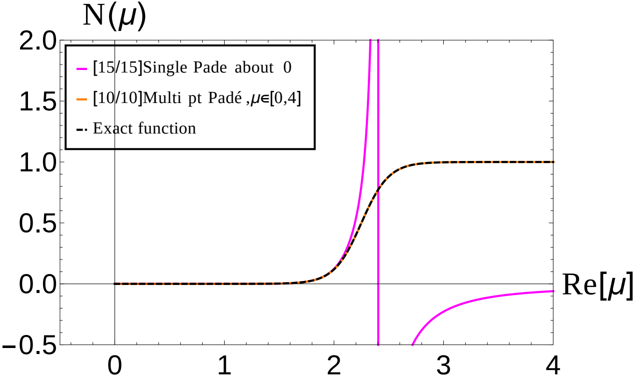
Roughly follow this procedure:

1. What transition are you interested in?
2. How should the singularities scale?
3. Lattice calculations at multiple, pure imaginary μ_B.
4. Estimate singularities with multi-point Padé.
5. Does scaling match expectation?
6. Analytically continue results to $\mu_B \in \mathbb{R}$.

Next: Why we trust it.

Test: 1-d Thirring model[16,17]

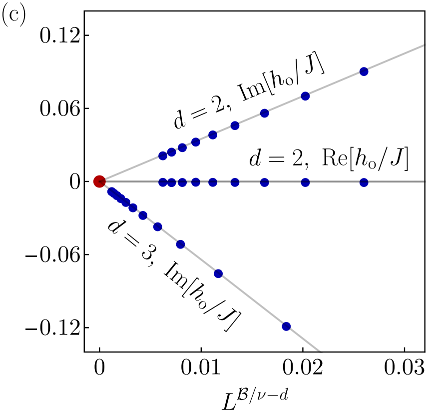
Number density $N(\mu)$ can be worked out exactly.

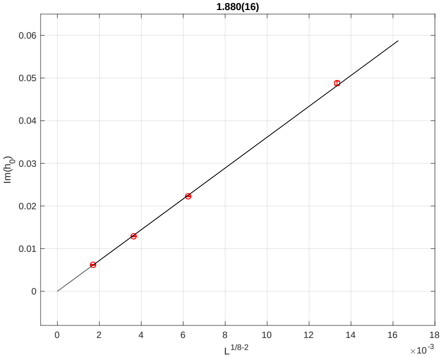


Multi-point captures the exact $N(\mu)$ well, outperforms single point.

[16] P. Dimopoulos et al., Phys. Rev. D, 105.3, 034513 (2022).
[17] F. Di Renzo, S. Singh, and K. Zambello, Phys. Rev. D, 103.3, 034513 (2021).

Test: 2-d Ising model[18,19]





Reproduces correct scaling and critical exponents extremely well.

[18] A. Deger and C. Flindt, Phys. Rev. Research, 1.2, 023004 (2019).
[19] F. Di Renzo and S. Singh, PoS(LATTICE2022)148, (2023).

Test: The Roberge-Weiss transition[21]

$\mathcal{Z}_{\rm QCD}$ at $\hat{\mu}_f = i\hat{\mu}_I$ has $\mathbb{Z}_3$ periodicity

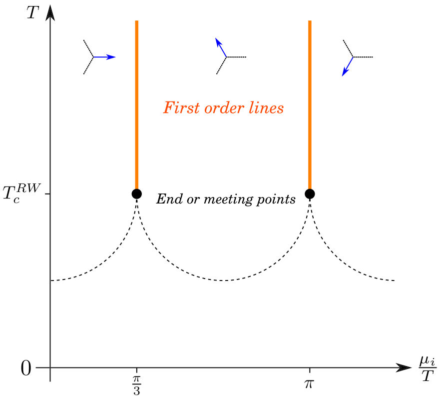
$$\hat{\mu}_I \to \hat{\mu}_I + 2\pi n/3$$

with $\hat{\mu} \equiv \mu/T$. First order lines separate phases distinguished by Polyakov loop

$$P \sim \sum_{\vec{x}} \mathrm{tr} \prod_{\tau} U_4(\vec{x}, \tau).$$

Endpoint in 3-d, $\mathbb{Z}_2$ universality class. Critical exponents[20]:

$$\beta = 0.3264, \qquad \delta = 4.7898$$



[20] S. El-Showk et al., J Stat Phys, 157.4-5, 869–914 (2014).
[21] F. Cuteri et al., Phys. Rev. D, 106.1, 014510 (2022).

Test: The Roberge-Weiss transition[22,23]

Lattice setup:

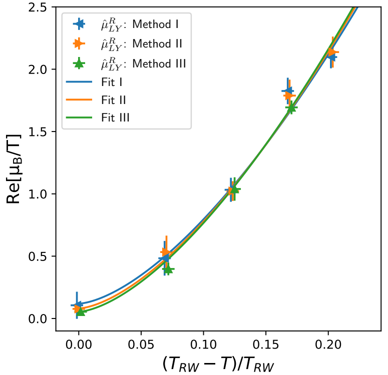
- 2+1 dynamical HISQ quarks
- m_s/m_l fixed to physical value
- $N_\tau = 4$, 6 with $N_s/N_\tau = 6$

$$h \sim \hat{\mu}_B - i\pi \qquad t \sim T - T_{\text{RW}}$$

$$z = th^{-1/\beta\delta} \qquad z_c = |z_c| e^{\pm i\pi/2\beta\delta}$$

$$\Rightarrow \quad \operatorname{Re}\hat{\mu}_{\text{LY}} = \pm\pi\left(\frac{z_0}{|z_c|}\right)^{\beta\delta}$$

Taking $|z_c| = 2.43$ yields $9.1 \lesssim z_0 \lesssim 9.4$.

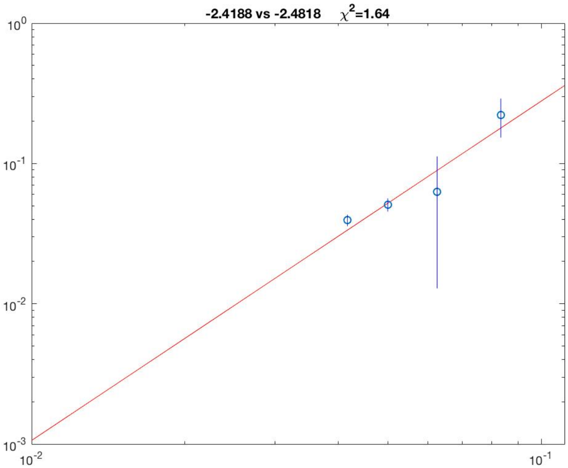


Taking $T_{\text{RW}}^{N_\tau=4} = 201.4$ MeV yields $\beta\delta \approx 1.5635$, compare 1.563495(15).

Prelim: $T_{\text{RW}} = 211.1(3.1)$ MeV, compare 208(5) MeV.

[22] C. Bonati et al., Phys. Rev. D, 93.7, 074504 (2016).
[23] G. Johnson et al., Phys. Rev. D, 107.11, 116013 (2023).

Test: Roberge-Weiss FSS



FSS scaling of $\mathrm{Re}\,\hat{\mu}_{\mathrm{LY}}$ near RW transition reasonably captured.

Toward the CEP

Assuming multi-point Padé reliable, turn attention to CEP. Also in 3-d, $\mathbb{Z}_2$ universality class, so $\beta\delta \approx 1.5$. Exact mapping to Ising not yet known. Linear ansatz:

$$t = \alpha_t \Delta T + \beta_t \Delta\mu_B$$
$$h = \alpha_h \Delta T + \beta_h \Delta\mu_B,$$

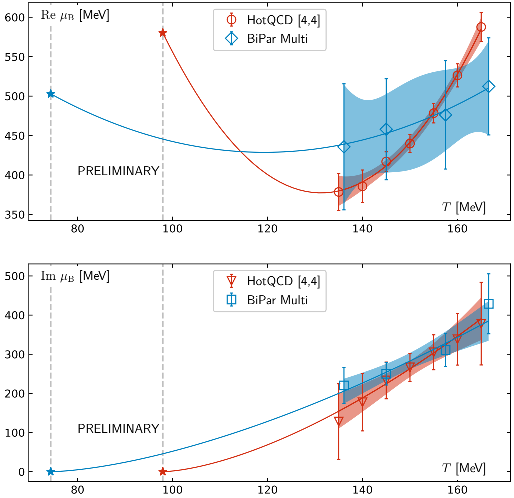
where $\Delta T \equiv T - T^{\text{CEP}}$ and $\Delta\mu_B \equiv \mu_B - \mu_B^{\text{CEP}}$, which leads to[24]

$$\mu_{\text{LY}} = \mu_B^{\text{CEP}} + c_1 \Delta T + i c_2 |z_c|^{-\beta\delta} \Delta T^{\beta\delta} + c_3 \Delta T^2 + \mathcal{O}\left(\Delta T^3\right).$$

Expectation from lattice[25]: $\mu_B^{\text{CEP}}/T^{\text{CEP}} \gtrsim 3$.

[24] M. A. Stephanov, Phys. Rev. D, 73.9, 094508 (2006).
[25] D. Bollweg et al., Phys. Rev. D, 105.7, 074511 (2022).

Toward the CEP: Single-point and multi-point



Some comments:

- Must propagate fit uncertainties
- Red: smaller N_s/N_τ
- Red: $N_\tau = 8$
- Blue: $N_\tau = 6$
- Blue: Need lower T

Rough suggestion of CEP:
$T \sim 85$ MeV $\quad \mu_B \sim 550$ MeV

Toward the CEP: Evaluation of rough estimate

$T \sim 85$ MeV $\mu_B \sim 550$ MeV

- $T < T_c \approx 130$ MeV[26]
- $\mu_B/T \sim 6.5$ is well outside apparent convergence radius

Year	Method	T^{CEP} [MeV]	μ_B^{CEP} [MeV]	$\mu_B^{\text{CEP}}/T^{\text{CEP}}$
2023	CP+LQCD[27]	≈ 100	≈ 580	≈ 5.8
2023	BHE[28]	101-108	560-625	≈ 5.7
2021	DSE[29]	117	600	5.13
2021	DSE[30]	109	610	5.59
2020	fRG[31]	107	635	5.54

[26] H.-T. Ding et al., Phys. Rev. Lett. 123.6, 062002 (2019).
[27] G. Basar, 2312.06952, (2023).
[28] M. Hippert et al., 2309.00579, (2023).
[29] P. J. Gunkel and C. S. Fischer, Phys. Rev. D, 104.5, 054022 (2021).
[30] F. Gao and J. M. Pawlowski, Phys. Lett. B, 820, 136584 (2021).
[31] W.-j. Fu, J. M. Pawlowski, and F. Rennecke, Phys. Rev. D, 101.5, 054032 (2020).

Summary and Outlook

- Multi-point Padé tested in a variety of situations
- Possible indication of CEP around $T \sim 85$ MeV, $\mu_B \sim 550$ MeV
- In progress: Refinement of CEP estimate strategy
- In progress: Continuum limit extrapolation
- In progress: Examine chiral transition

Thanks for your attention.