

QCD material parameters from the lattice

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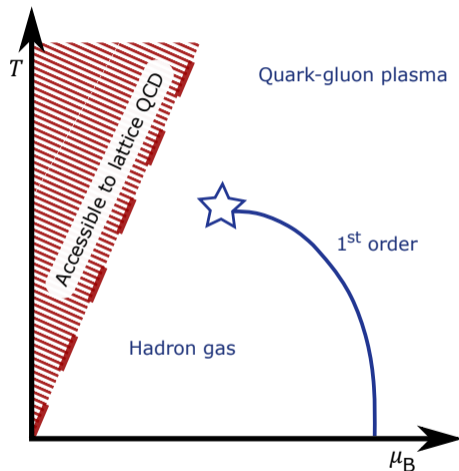
Motivation: Broad strokes

Broadly interested in **phase diagram** of strongly interacting systems.

At high enough temperatures and/or densities, hadrons dissociate to quark-gluon plasma.

Relevant to several systems:

- ▶ Early universe
- ▶ Neutron stars (NS)
- ▶ Heavy ion collisions (HIC)



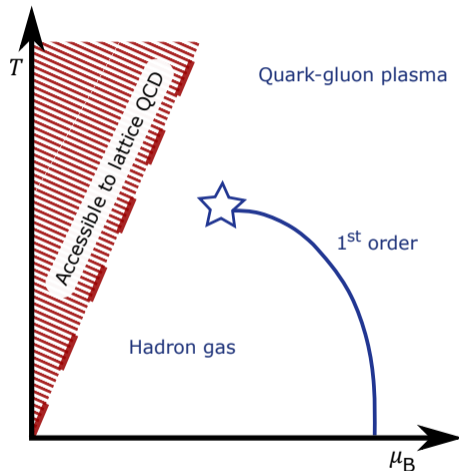
Motivation: What lattice can do

$$\begin{aligned}\langle O \rangle &\sim \int d\bar{\psi} d\psi e^{-\bar{\psi} D \psi} \int dU e^{-S(U)} O(U) \\ &= \int \det D \int dU e^{-S(U)} O(U)\end{aligned}$$

Complication (**sign problem**):

- ▶ $\det D \in \mathbb{R}$ when $\mu = 0$
- ▶ But if $\mu \neq 0$, it is complex...
- ▶ Can use tricks:

$$\frac{p}{T^4} = \sum_{i,j,k=0}^{\infty} A_{ijk} \left(\frac{\mu_B}{T}\right)^i \left(\frac{\mu_Q}{T}\right)^j \left(\frac{\mu_S}{T}\right)^k$$



Want to learn about composition and properties of strongly interacting systems. One way is through **material parameters** like **isentropic sound speed**

$$c_s^2 \equiv \left(\frac{\partial p}{\partial \epsilon} \right)_{s/n_B}$$

Material parameters give physical intuition how strongly interacting matter deforms, expands, etc. Can have more utility: c_s^2 particularly useful, e.g.

- ▶ related to fireball expansion rate^a
- ▶ has “soft point” (minimum) near crossover
- ▶ check if NS centers contain hadronic d.o.f.^b

^aJ. D. Bjorken, Phys. Rev. D, 27, 140–151 (1983).

^bI. Tews et al., Astrophys. J., 860.2, 149 (2018).

GOAL: compute material parameters at $\mu_B > 0$.

But only have *direct* access to $\mu_B = 0$ on the lattice. Commonly played game:

1. Write p/T^4 as **Taylor expansion** in μ_i/T
2. Derive material parameters from p/T^4 using **thermodynamics**
3. **Measure** Taylor coefficients on lattice
4. Compare against **HRG** for $T < T_{pc}$ (**crossover temp** ~ 156 MeV)
5. Compare against **ideal gas** for $T \gg T_{pc}$

Lattice pressure

For convenience, $\hat{X} \equiv XT^{-k}$ with k s.t. \hat{X} dimensionless (e.g. $\hat{\mu} = \mu/T$)

Dealing with 3 chemical potentials $\hat{\mu}_B, \hat{\mu}_Q, \hat{\mu}_S$

To make contact with T - $\hat{\mu}_B$ plane, need to eliminate 2 independent variables

Can **impose external constraint**, e.g.

1. $\hat{\mu}_Q = \hat{\mu}_S = 0$
2. $n_S = 0, n_Q/n_B = 0.4$ (RHIC-like initial conditions, collide Au nuclei)
3. $n_S = 0, n_Q/n_B = 0.5$ (isospin-symmetric; yields $\hat{\mu}_Q = 0$)

and think of **expansions in $\hat{\mu}_B$ only**:

$$\hat{p} = \sum_{i,j,k=0}^{\infty} \frac{\chi_{ijk}^{BQS}}{i!j!k!} \hat{\mu}_B^i \hat{\mu}_Q^j \hat{\mu}_S^k, \quad \chi_2^B \equiv \frac{\partial^2 \hat{p}}{\partial \hat{\mu}_B^2} \quad \rightarrow \quad \hat{p} = \sum_{k \text{ even}} \frac{P_k(T)}{k!} \hat{\mu}_B^k$$

Hadron resonance gas pressure

Non-interacting, quantum, relativistic gas eventually gives (single species)

$$\hat{p}^{\text{HRG}} = \frac{m^2 g}{2\pi^2 T^2} \sum_{k=1}^{\infty} \frac{\eta^{k+1} z^k}{k^2} K_2 \left(\frac{mk}{T} \right), \quad z \equiv e^{\hat{\mu}_B B + \hat{\mu}_Q Q + \hat{\mu}_S S},$$

with K_2 modified Bessel function 2nd kind. HRG:

- ▶ Assume such gas where hadrons and resonances only d.o.f.
- ▶ Hence valid up to $\sim T_{\text{pc}}$
- ▶ Sum over all such states, each with g_i , m_i , etc.
- ▶ K_2 exponentially suppressed, so can keep few terms

Ideal, massless gas of up, down, and strange quarks:

$$\hat{p}^{\text{id}} = \frac{19\pi^2}{36} + \frac{1}{2} (\hat{\mu}_u^2 + \hat{\mu}_d^2 + \hat{\mu}_s^2) + \frac{1}{4\pi^2} (\hat{\mu}_u^4 + \hat{\mu}_d^4 + \hat{\mu}_s^4),$$

which can be rewritten using

$$\mu_u = \frac{1}{3}\mu_B + \frac{2}{3}\mu_Q$$

$$\mu_d = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q$$

$$\mu_s = \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q - \mu_S.$$

Extracting a material parameter

EXAMPLE: $c_s^2 \sim \left(\frac{\partial p}{\partial \epsilon}\right)_s$ with only one μ .

$$dp = \frac{\partial p}{\partial T} dT + \frac{\partial p}{\partial \mu} d\mu \quad (1)$$

$$d\epsilon = \frac{\partial \epsilon}{\partial T} dT + \frac{\partial \epsilon}{\partial \mu} d\mu \quad (2)$$

$$ds = \frac{\partial s}{\partial T} dT + \frac{\partial s}{\partial \mu} d\mu \quad (3)$$

Think of each $O(T, \mu)$. But also $p(\epsilon, s)$:

$$\begin{aligned} dp &= \frac{\partial p}{\partial \epsilon} d\epsilon + \frac{\partial p}{\partial s} ds \\ &= c_s^2 d\epsilon + \frac{\partial p}{\partial s} ds \end{aligned} \quad (4)$$

Hence use (2) and (3) to eliminate $d\mu$ and dT in favor of $d\epsilon$ and ds in (1):

$$c_s^2 = \frac{n^2 \frac{\partial s}{\partial T} - 2sn \frac{\partial s}{\partial \mu} + s^2 \frac{\partial n}{\partial \mu}}{(\epsilon + p) \left(\frac{\partial s}{\partial T} \frac{\partial n}{\partial \mu} - \frac{\partial s}{\partial \mu} \frac{\partial n}{\partial T} \right)},$$

where

$$s = \frac{\partial p}{\partial T}, \quad n = \frac{\partial p}{\partial \mu}$$

Everything in terms of μ -,
 T -derivatives of p !

What material parameters we look at

We have μ_B, μ_Q, μ_S .

- ▶ Two more chemical potentials \Rightarrow fix more quantities
- ▶ Eventually recast everything in terms of intensive quantities

$$\begin{aligned}c_s^2 &= \left(\frac{\partial p}{\partial \epsilon} \right)_{s/n_B, r, n_S/n_B} & \kappa_s &= -\frac{1}{V} \left(\frac{\partial V}{\partial p} \right)_{S, \vec{N}} = \frac{1}{n_B} \left(\frac{\partial n_B}{\partial p} \right)_{s/n_B, r, n_S/n_B} \\c_T^2 &= \left(\frac{\partial p}{\partial \epsilon} \right)_{T, r, n_S/n_B} & \kappa_T &= -\frac{1}{V} \left(\frac{\partial V}{\partial p} \right)_{T, \vec{N}} = \frac{1}{n_B} \left(\frac{\partial n_B}{\partial p} \right)_{T, r, n_S/n_B} \\C_V &= \frac{T}{V} \left(\frac{\partial s}{\partial T} \right)_{V, \vec{N}} & &= T \left(\frac{\partial s}{\partial T} \right)_{n_B, r, n_S/n_B}\end{aligned}$$

An example

For instance, the **isothermal compressibility** comes out to be

$$\kappa_T = \left(\frac{(\chi_{11}^{BS})^2 \chi_2^Q - 2\chi_{11}^{BQ} \chi_{11}^{BS} \chi_{11}^{QS} + \chi_2^B (\chi_{11}^{QS})^2 + (\chi_{11}^{BQ})^2 \chi_2^S - \chi_2^B \chi_2^Q \chi_2^S}{n_B^2 b_{B2}} \right)_{T,r,n_S/n_B=0}$$

Here $\chi_{ijk}^{BQS}(\vec{\mu}, T)$ and $b_{B2}(r, \chi_{ijk}^{BQS})$.

κ_T diverges as $\mu_B \rightarrow 0$.

Ideal gas limit at $\vec{\mu} = 0$:

$$\left. \frac{n_B^2 \kappa_T^{\text{id}}}{T^2} \right|_{\vec{\mu}=0} = \frac{2}{9}$$

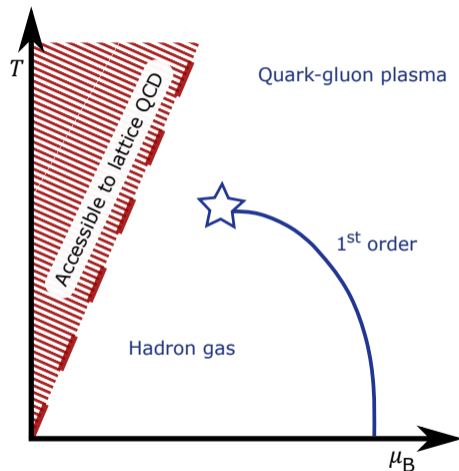
More uses of material parameters

Discussed already some uses of c_s^2 . Also

- ▶ Look for critical point
 - c_s^2 hits 0
 - $\kappa_T \sim |T - T^{\text{CEP}}|^{-\gamma}$, $\gamma \approx 1.23$
 - $C_V \sim |T - T^{\text{CEP}}|^{-\alpha}$, $\alpha \approx 0.11$
 - location^{a,b}: $T^{\text{CEP}} \lesssim 110$, $\mu_B^{\text{CEP}} \gtrsim 420$ MeV
- ▶ κ_T relates to n_B fluctuations
- ▶ C_V relations to T fluctuations

^aD. A. Clarke et al., PoS, LATTICE2023, 168 (2024).

^bJ. Goswami et al., QM2023, (Jan. 2024).



Some context and lattice setup

Related studies from the past, for instance $\hat{\mu}_B = 0^{1,2}$ and $\hat{\mu}_B > 0^{3,4}$.

There you can find c_s^2 and C_V . This study:

- ▶ Taylor series up to 8th order; converges well at least for $\hat{\mu}_B \lesssim 2^{5,6}$.
- ▶ Set physical $m_s/m_l = 27$.
- ▶ Focus⁷ on $n_S = 0$ and $r \equiv n_Q/n_B = 0.5$.
- ▶ (Seemingly) first lattice determinations κ_T and c_T^2 .

¹A. Bazavov et al., Phys. Rev. D, 90, 094503 (2014).

²S. Borsanyi et al., Phys. Lett. B, 730, 99–104 (2014).

³A. Bazavov et al., Phys. Rev. D, 95.5, 054504 (2017).

⁴S. Borsanyi et al., JHEP, 10, 205 (2018).

⁵D. Bollweg et al., Phys. Rev. D, 105.7, 074511 (2022).

⁶Strictly speaking, convergence radius depends on temperature.

⁷Results at $r = 0.4$ and $r = 0.5$ are similar.

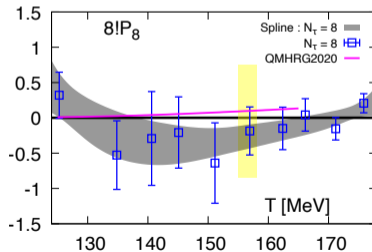
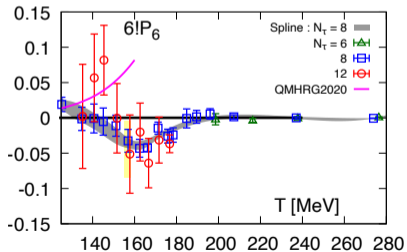
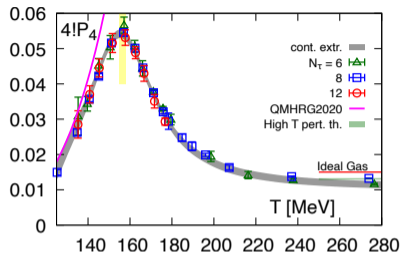
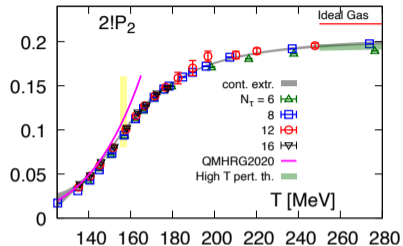
Some context and lattice setup

- ▶ $N_f = 2 + 1$ HISQ ensembles
- ▶ Set scale with f_K
- ▶ Approx. 1.5M, 300k, 22k configs for $N_\tau = 8, 12, 16$, respectively
- ▶ Continuum extrapolated for $\mathcal{O}(\hat{\mu}_B^2)$ and $\mathcal{O}(\hat{\mu}_B^4)$ coefficients:
 - $125 \leq T \leq 145$ MeV: HRG-inspired ansatz

$$f(T) = A \left(\frac{B}{T} \right)^{3/2} e^{-B/T} \left(1 + d_1 T + d_2 T^2 + \frac{c}{N_\tau^2} \right)$$

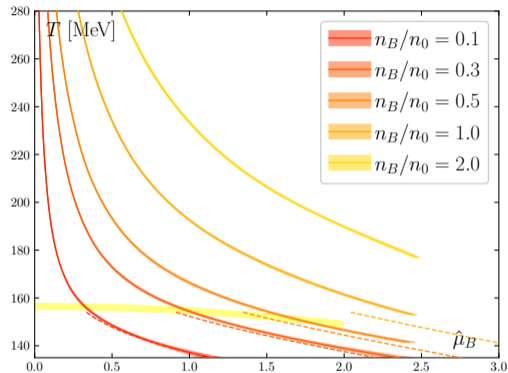
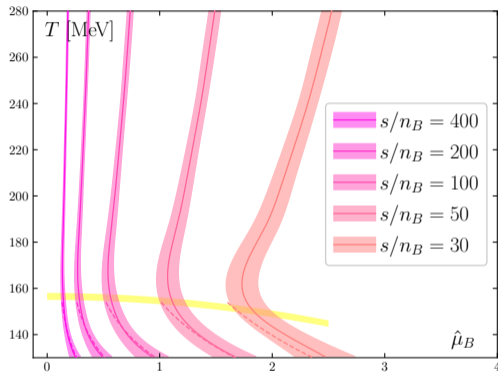
- $135 \leq T \leq 175$: spline
- $T \geq 175$ MeV: polynomial
- stitch extrapolations from three regions together with spline

HotQCD pressure coefficients⁸



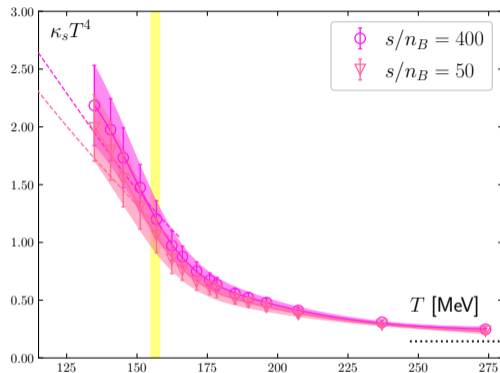
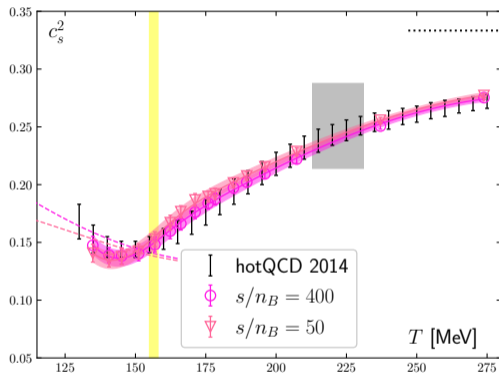
⁸D. Bollweg et al., Phys. Rev. D, 108.1, 014510 (2023).

Lines of constant physics $n_S = 0$, $n_Q/n_S = 0.5$



$$n_0 = 0.16 \text{ fm}^{-3}$$

Results: Isentropic observables^{9,10}

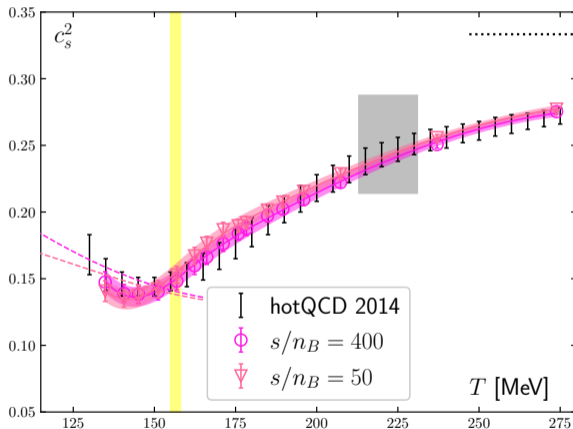


$$\kappa_s = \frac{1}{c_s^2(\epsilon + p - \mu_Q n_Q - \mu_S n_S)}$$

⁹F. G. Gardim et al., Nature Phys. 16.6, 615–619 (2020).

¹⁰A. Bazavov et al., Phys. Rev. D, 90, 094503 (2014).

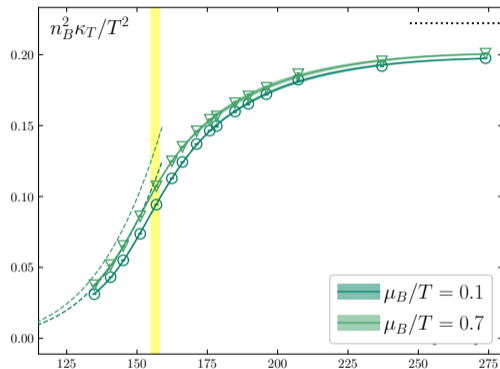
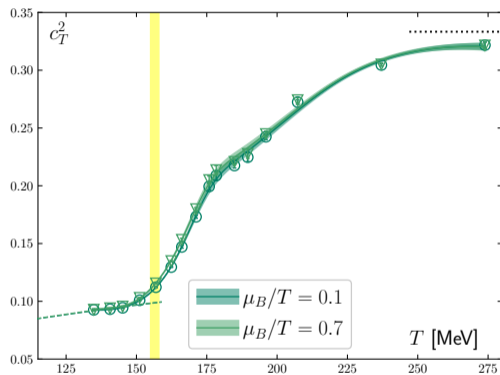
Results: Isentropic observables



In the surveyed range ($\hat{\mu}_B \lesssim 3$):

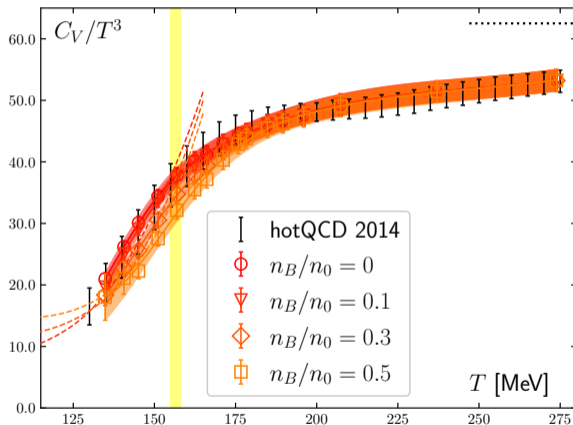
- ▶ Dip near T_{pc}
- ▶ Large s/n_B shows good agreement with $\vec{\mu} = 0$
- ▶ Minimal dependence on $\vec{\mu}$
- ▶ Agreement with Gardim *et al.* extraction
- ▶ No violation of conformal limit

Results: Isothermal observables



c_T^2 has no dip
No critical behavior (p converges)

Results: Isovolumetric specific heat



In the surveyed range ($\hat{\mu}_B \lesssim 3$):

- ▶ $n_B = 0$ agrees with $\vec{\mu} = 0$
- ▶ High T sees no $\vec{\mu}$ dependence
- ▶ Mild dependence near T_{pc}
- ▶ No critical behavior

- ▶ Derived formulae for material parameters that obey several theoretical cross-checks
- ▶ Have lattice data for c_s^2 , c_T^2 , κ_s , κ_T , and C_V for $\hat{\mu}_B \lesssim 3$
- ▶ (Seemingly) first lattice determinations of c_T^2 and κ_T
- ▶ With exception of κ_T , observables seem to have weak $\hat{\mu}_B$ dependence
- ▶ Want to try fixed p observables but these seem challenging

Thanks for your attention.