Searching for the QCD critical point using Lee-Yang edge singularities

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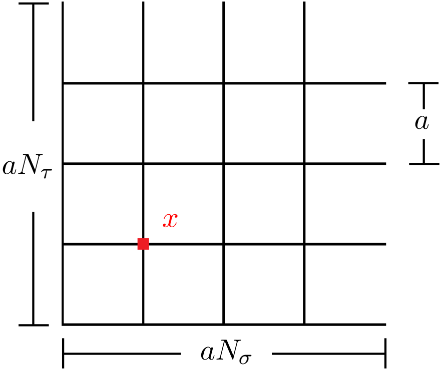
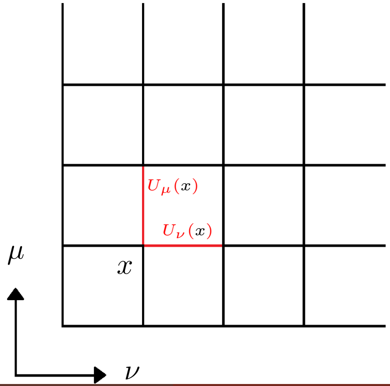
UIC Seminar, 3 Apr 2024

What lattice people do (no fermions)

- 4D space-time with Euclidean metric and periodic BCs
- Sites $x = (an_1, an_2, an_3, an_4)$
- Regularization through lattice spacing a
- UV cutoff $\sim 1/a$; IR cutoff $\sim 1/aN$



- μ and ν label directions
- Link variables $U_\mu(x) = e^{-aA_\mu(x)} \in \mathrm{SU}(3)$ on links
- Configuration: Snapshot of all $4 \times N_\sigma^3 \times N_\tau$ links



Path integrals

Want expected value of operator O.

- QFT expectation values:

$$\langle O \rangle \sim \int \mathcal{D}U \, e^{iS(U)} \, O(U)$$

- Lattice QCD expectation values:

$$\langle O \rangle \sim \int \mathcal{D}U \, e^{-S(U)} \, O(U)$$

- Achieved through Wick rotation

$$t \to i\tau$$

- Hence our Metric goes from Minkowski to Euclidean

MCMC

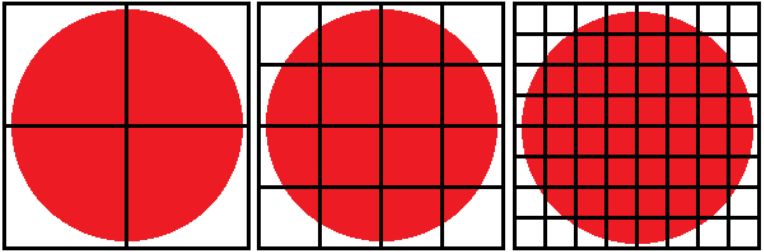
- Markov Chain Monte Carlo basic idea:
  - Each configuration generated depending on last one only
  - Accept new configuration with probability $\min\{1, e^{-\Delta S}\}$
  - Create a time series of measurements O_n of O
- The estimator for $\langle O \rangle$ on the lattice is

$$\bar{O} = \frac{1}{N_{\text{conf}}} \sum_{n=1}^{N_{\text{conf}}} O_n$$

Continuum limit

- Continuum limit: $a \to 0$
- Must also increase the number of sites
- Carry out a fit, usually need 3 or more spacings:

$$O(a) = O^{\text{cont.}} + a^2 O_0$$



A little more detail when there's fermions

$$\langle O\rangle \sim \int \mathrm{d}\bar{\psi}\,\mathrm{d}\psi\, e^{-\bar{\psi}D\psi} \quad \mathrm{d}U\, e^{-S(U)}\, O(U)$$
$$= \int \det D \qquad\qquad \mathrm{d}U\, e^{-S(U)}\, O(U)$$

Complication:

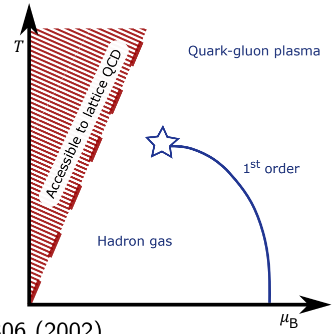
- $\det D \in \mathbb{R}$ when $\mu = 0$
- But if $\mu \neq 0$, it is complex...

The infamous problem

Trick: μ_B pure imaginary avoids sign problem; can analytically continue to $\mu_B \in \mathbb{R}$[1,2].

Trick: Expand pressure P/T^4 in μ_B/T[3,4].

The latter is getting a bit too pricey. Popularity of resummation schemes[5,6,7,8].



[1] P. de Forcrand and O. Philipsen, Nuclear Physics B, 642.1-2, 290–306 (2002).
[2] M. D'Elia and M.-P. Lombardo, Phys. Rev. D, 67.1, 014505 (2003).
[3] C. R. Allton et al., Phys. Rev. D, 66.7, 074507 (2002).
[4] R. V. Gavai and S. Gupta, Phys. Rev. D, 68.3, 034506 (2003).
[5] S. Borsányi et al., Phys. Rev. Lett. 126.23, 232001 (2021).
[6] D. Bollweg et al., Phys. Rev. D, 105.7, 074511 (2022).
[7] S. Mitra, P. Hegde, and C. Schmidt, Phys. Rev. D, 106.3, 034504 (2022).
[8] S. Mondal, S. Mukherjee, and P. Hegde, Phys. Rev. Lett. 128.2, 022001 (2022).

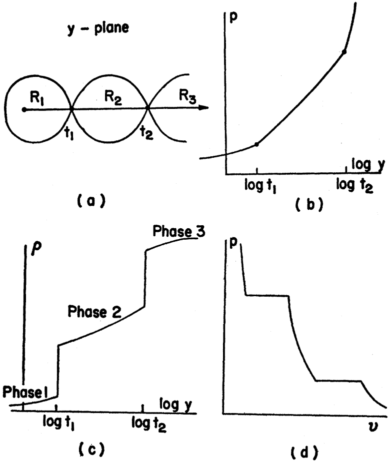
Lee-Yang theorem

Works where $\log \mathcal{Z}_{\rm QCD}$ is free of singularities.

Lee-Yang theorem[9]: Zeroes of the partition function that approach the real axis as $V \to \infty$ correspond to phase transitions.

Intuition: Indications of non-analyticities in P

- may hint at phase transitions
- or singularities in $\mathbb{C}$
- constrain validity of Taylor series



[9] C. N. Yang and T. D. Lee, Phys. Rev. 87.3, 404–409 (1952).

Ising: Generically have branch cuts on imaginary axis. (Pinch real axis at T_c.)

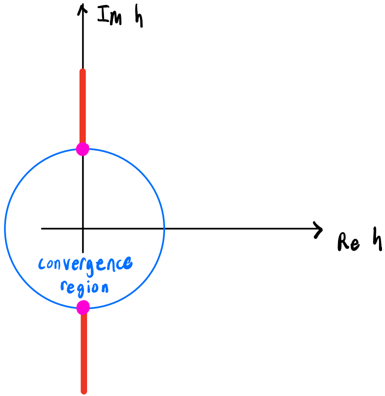
Lee-Yang edge (LYE): The singularities closest to real axis.

Above T_c, LYE nearest singularity to the origin.

LYE position fixed at

$$z_c = |z_c| e^{\pm i\pi/2\beta\delta}$$

with $z \equiv t h^{-1/\beta\delta}$ and critical exponents β, δ.



Padé approximants

Want detailed information about singularities $\Rightarrow$ **rational functions**,

$$R_n^m(x) \equiv \frac{\sum_{i=0}^m a_i x^i}{1+\sum_{j=1}^n b_j x^j}.$$

Singularities captured or mimicked by zeros in denominator.

Let f have a formal Taylor series

$$f(x) = \sum_{k=0}^{\infty} c_k x^k.$$

Padé approximant of order $[m,n]$: R_n^m with coefficients so that it equals the Taylor series up to order $m+n$. Gives relationship between coefficients a_i, b_j, c_k.

Padé approximants

Things to think about with Padé:

- Theorem: Unique when it exists
- Theorem: $[m, n]$ converges to f exactly as $m \to \infty$ when f has pole of order n
- Other properties deduced from numerical experiments
- Limited by number of known Taylor coefficients
- Only have up to 8^{th} order[10,11] for $\log \mathcal{Z}_{\text{QCD}}$; difficultly far greater for higher orders[12]

[10] S. Borsanyi et al., J. High Energ. Phys. 2018.10, 205 (2018).
[11] D. Bollweg et al., Phys. Rev. D, 108.1, 014510 (2023).
[12] Computational requirements of HotQCD EoS exceed 2000 GPU-years and 2.4 PB.

Multi-point Padé approximants

Padé approximants you get by demanding[13]

$$R_n^m(x) = f^{m+n}(x) \equiv \sum_{i=0}^{m+n} c_k x^k.$$

Say we know Taylor series up to some order s. The Multi-point Padé is the R_n^m satisfying

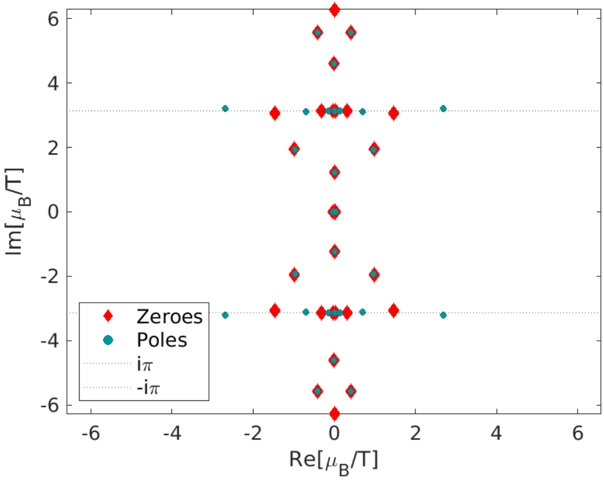
$$\left.\frac{\mathrm{d}^l R_n^m}{\mathrm{d}x^l}\right|_{x_i} = \left.\frac{\mathrm{d}^l f}{\mathrm{d}x^l}\right|_{x_i}$$

for N points x_i, $0 \leq l < s-1$. Some pros/cons:

- Need fewer Taylor coefficients!
- Less seems to be known about them...

[13] One expects corresponding relationships among derivatives of R and f.

Extracting a LYE[14]



[14] P. Dimopoulos et al., Phys. Rev. D, 105.3, 034513 (2022).

The strategy

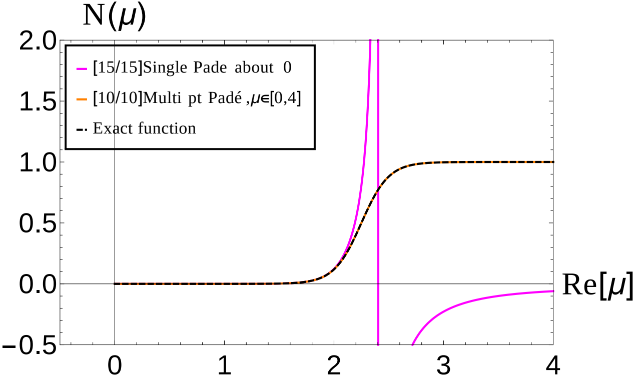
Roughly follow this procedure:

1. What transition are you interested in?
2. How should the singularities scale?
3. Lattice calculations at multiple, pure imaginary μ_B.
4. Estimate singularities with multi-point Padé.
5. Does scaling match expectation?
6. Analytically continue results to $\mu_B \in \mathbb{R}$.

Next: Why we trust it.

Test: 1-d Thirring model[15,16]

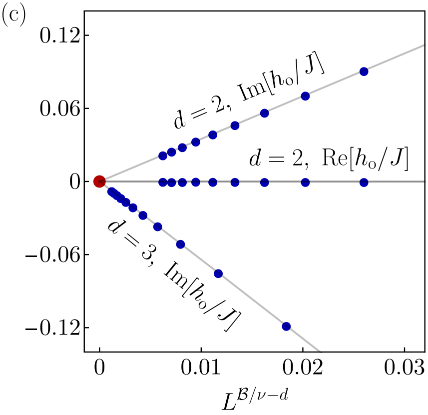
Number density $N(\mu)$ can be worked out exactly.

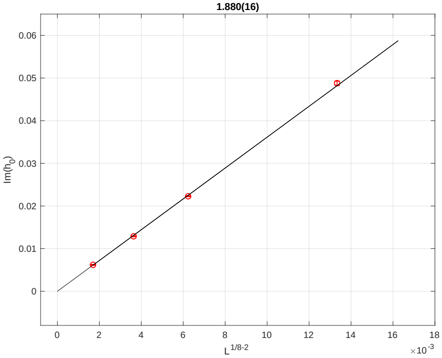


Multi-point captures the exact $N(\mu)$ well, outperforms single point.

[15] P. Dimopoulos et al., Phys. Rev. D, 105.3, 034513 (2022).
[16] F. Di Renzo, S. Singh, and K. Zambello, Phys. Rev. D, 103.3, 034513 (2021).

Test: 2-d Ising model[17,18]





Reproduces correct FSS scaling for h^{LY} at T_c.

[17]A. Deger and C. Flindt, Phys. Rev. Research, 1.2, 023004 (2019).
[18]F. Di Renzo and S. Singh, PoS(LATTICE2022)148, (2023).

Test: The Roberge-Weiss transition[20]

$\mathcal{Z}_{\rm QCD}$ at $\hat{\mu}_f = i\hat{\mu}_I$ has $\mathbb{Z}_3$ periodicity

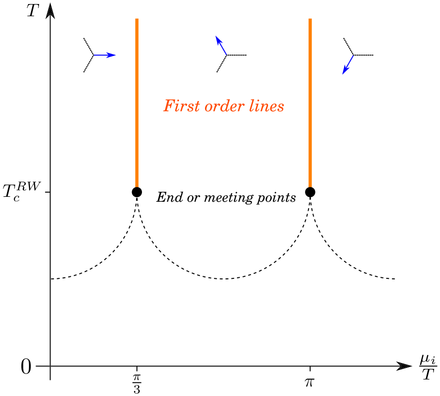
$$\hat{\mu}_I \to \hat{\mu}_I + 2\pi n/3$$

with $\hat{\mu} \equiv \mu/T$. First order lines separate phases distinguished by Polyakov loop

$$P \sim \sum_{\vec{x}} \mathrm{tr} \prod_{\tau} U_4(\vec{x},\tau).$$

Endpoint in 3-d, $\mathbb{Z}_2$ universality class. Critical exponents[19]:

$$\beta = 0.3264, \qquad \delta = 4.7898$$



[19] S. El-Showk et al., J Stat Phys, 157.4-5, 869–914 (2014).
[20] F. Cuteri et al., Phys. Rev. D, 106.1, 014510 (2022).

Test: The Roberge-Weiss transition[21,22]

Lattice setup:

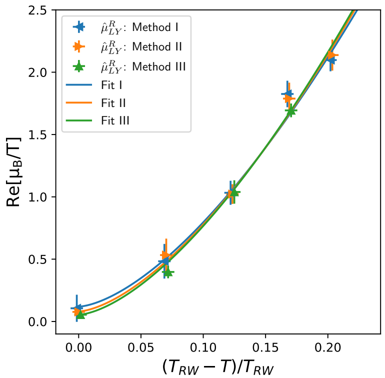
- 2+1 dynamical HISQ quarks
- m_s/m_l fixed to physical value
- $N_\tau = 4$, 6 with $N_s/N_\tau = 6$

$$h \sim \hat{\mu}_B - i\pi \qquad t \sim T - T_{\mathrm{RW}}$$

$$z = th^{-1/\beta\delta} \qquad z_c = |z_c| e^{\pm i\pi/2\beta\delta}$$

$$\Rightarrow \quad \mathrm{Re}\,\hat{\mu}_{\mathsf{LY}} = \pm\pi \left(\frac{z_0}{|z_c|}\right)^{\beta\delta}$$

Taking $|z_c| = 2.43$ yields $9.1 \lesssim z_0 \lesssim 9.4$.

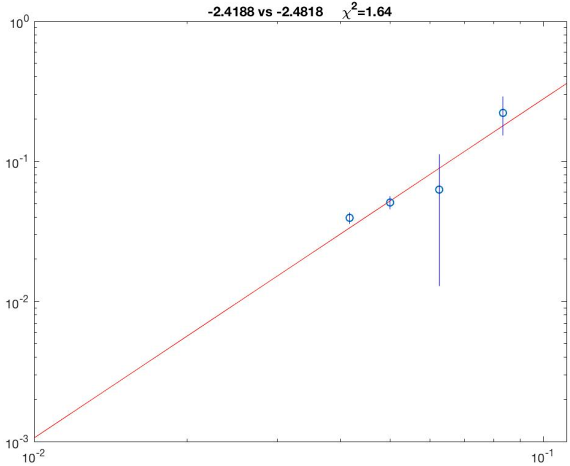


Taking $T_{\mathrm{RW}}^{N_\tau=4} = 201.4$ MeV yields $\beta\delta \approx 1.5635$, compare 1.563495(15).

Prelim: $T_{\mathrm{RW}} = 211.1(3.1)$ MeV, compare 208(5) MeV.

[21] C. Bonati et al., Phys. Rev. D, 93.7, 074504 (2016).
[22] G. Johnson et al., Phys. Rev. D, 107.11, 116013 (2023).

Test: Roberge-Weiss FSS[23]



FSS scaling of $\mathrm{Re}\,\hat{\mu}_{\mathsf{LY}}$ near RW transition reasonably captured.

[23]F. Di Renzo et al., PoS(LATTICE2023)169, (2024).

Toward the CEP

Assuming multi-point Padé reliable, turn attention to CEP. Also in 3-d, $\mathbb{Z}_2$ universality class, so $\beta\delta \approx 1.5$. Exact mapping to Ising not yet known. Linear ansatz:

$$t = \alpha_t \Delta T + \beta_t \Delta\mu_B$$
$$h = \alpha_h \Delta T + \beta_h \Delta\mu_B,$$

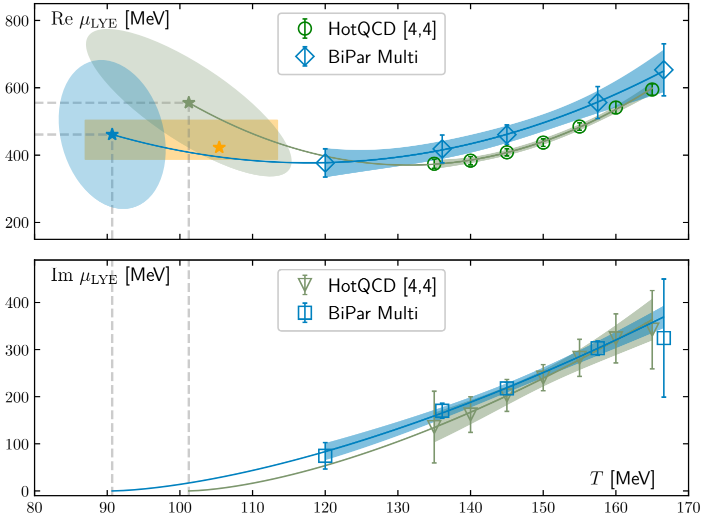
where $\Delta T \equiv T - T^{\text{CEP}}$ and $\Delta\mu_B \equiv \mu_B - \mu_B^{\text{CEP}}$, which leads to[24]

$$\mu_{\text{LY}} = \mu_B^{\text{CEP}} + c_1 \Delta T + i c_2 |z_c|^{-\beta\delta} \Delta T^{\beta\delta} + c_3 \Delta T^2 + \mathcal{O}\left(\Delta T^3\right).$$

Expectation from lattice[25]: $\mu_B^{\text{CEP}}/T^{\text{CEP}} \gtrsim 3$.

[24] M. A. Stephanov, Phys. Rev. D, 73.9, 094508 (2006).
[25] D. Bollweg et al., Phys. Rev. D, 105.7, 074511 (2022).

Toward the CEP: Single-point and multi-point



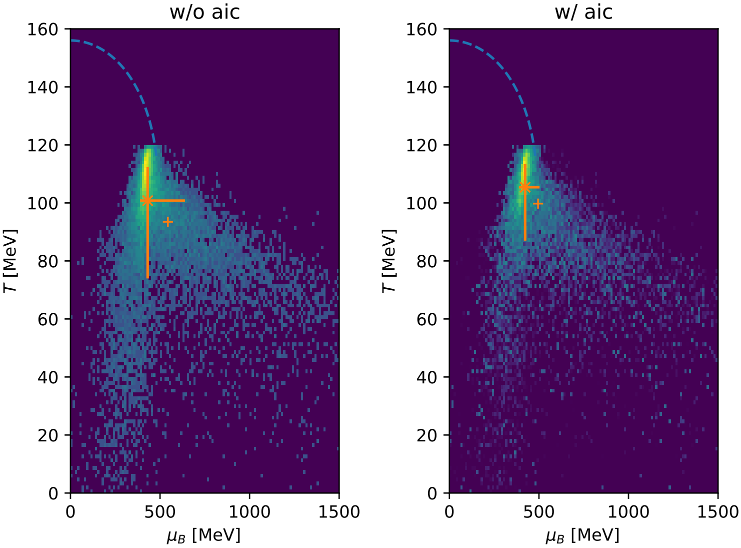
Some comments:

- Green: smaller box size
- Green: Finer lattice
- Green: Higher statistics
- Yellow box:

$$T^{\text{CEP}} = 105^{+8}_{-18}\ \text{MeV}$$

$$\mu_B^{\text{CEP}} = 423^{+80}_{-35}\ \text{MeV}$$

Toward the CEP: Distribution (yellow box)



Toward the CEP: Comparison

$$T^{\text{CEP}} = 105^{+8}_{-18}\ \text{MeV} \qquad \mu_B^{\text{CEP}} = 423^{+80}_{-35}\ \text{MeV}$$

- $T < T_c \approx 130$ MeV[26]
- $\mu_B^{\text{CEP}}/T^{\text{CEP}} \sim 4$ is outside apparent convergence radius

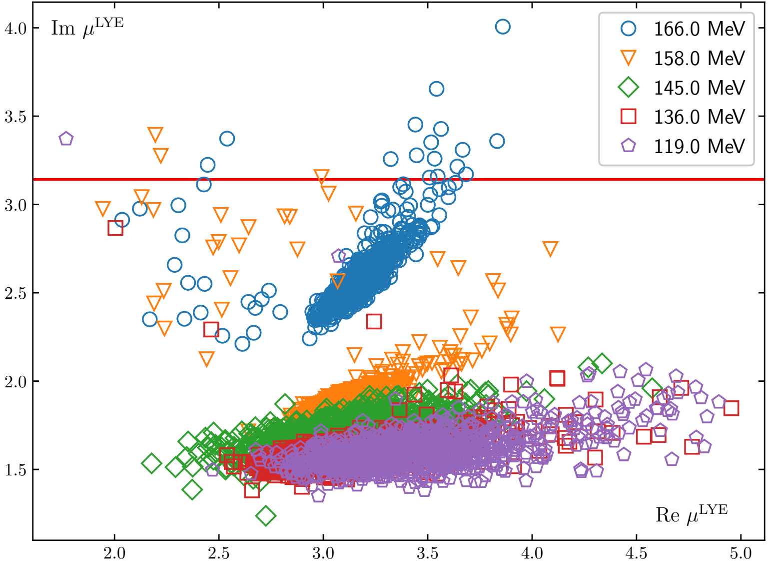
Year	Method	T^{CEP} [MeV]	μ_B^{CEP} [MeV]	$\mu_B^{\text{CEP}}/T^{\text{CEP}}$
2023	CP+LQCD[27]	≈ 100	≈ 580	≈ 5.8
2023	BHE[28]	101-108	560-625	≈ 5.7
2021	DSE[29]	117	600	5.13
2021	DSE[30]	109	610	5.59
2020	fRG[31]	107	635	5.54

[26] H.-T. Ding et al., Phys. Rev. Lett. 123.6, 062002 (2019).
[27] G. Basar, 2312.06952, (2023).
[28] M. Hippert et al., 2309.00579, (2023).
[29] P. J. Gunkel and C. S. Fischer, Phys. Rev. D, 104.5, 054022 (2021).
[30] F. Gao and J. M. Pawlowski, Phys. Lett. B, 820, 136584 (2021).
[31] W.-j. Fu, J. M. Pawlowski, and F. Rennecke, Phys. Rev. D, 101.5, 054032 (2020).

Summary and Outlook

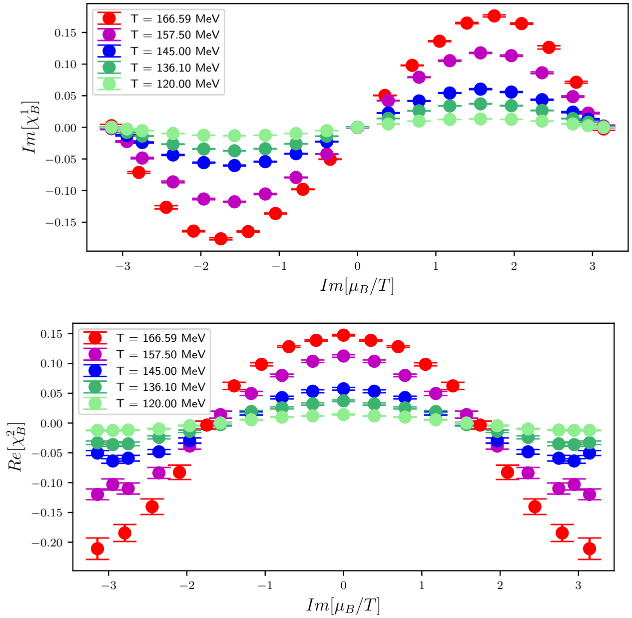
- Multi-point Padé tested in a variety of situations
- Indication of CEP around $T^{\text{CEP}} = 105^{+8}_{-18}$ MeV, $\mu_B^{\text{CEP}} = 423^{+80}_{-35}$ MeV
- In progress: Computations on finer lattices
- In progress: Examine chiral transition

Thanks for your attention.


Im $\mu^{\rm LYE}$
Re $\mu^{\rm LYE}$
166.0 MeV
158.0 MeV
145.0 MeV
136.0 MeV
119.0 MeV
4.0
3.5
3.0
2.5
2.0
1.5
2.0
2.5
3.0
3.5
4.0
4.5
5.0

β	T [MeV]	N_{conf}
6.170	166.6	1800
6.120	157.5	4780
6.038	145.0	5300
5.980	136.1	6840
5.850	120.0	24000

Table: Statistics for each ensemble used in this study. The last column gives approximate number of thermalized configurations per μ value. Quark masses are fixed to their physical value and $\mu_s = \mu_l$.


T = 166.59 MeV
T = 157.50 MeV
T = 145.00 MeV
T = 136.10 MeV
T = 120.00 MeV
$Im[\chi_B^1]$
$Im[\mu_B/T]$
T = 166.59 MeV
T = 157.50 MeV
T = 145.00 MeV
T = 136.10 MeV
T = 120.00 MeV
$Re[\chi_B^2]$
$Im[\mu_B/T]$