QCD material parameters from the lattice

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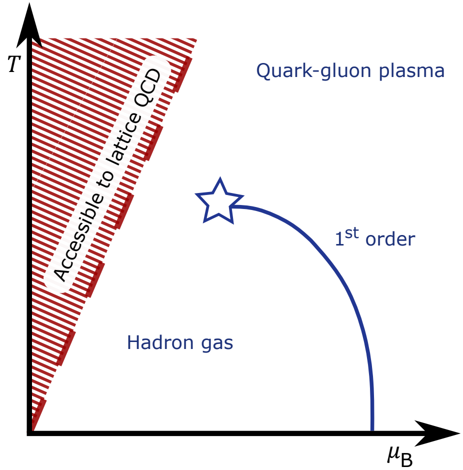
Motivation: Broad strokes

Broadly interested in **phase diagram** of strongly interacting systems.

At high enough temperatures and/or densities, hadrons dissociate to quark-gluon plasma.

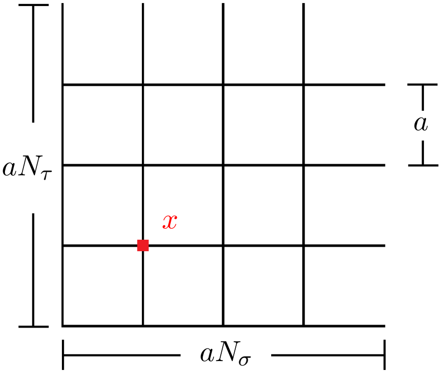
Relevant to several systems:

- Early universe
- Neutron stars (NS)
- Heavy ion collisions (HIC)



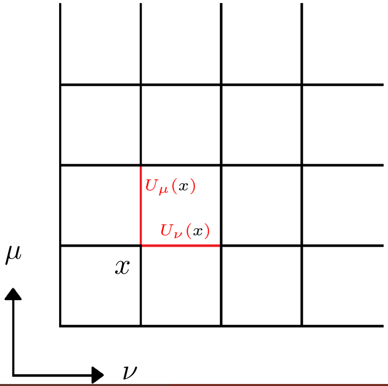
What lattice people do (no fermions)

- 4D space-time with Euclidean metric and periodic BCs
- Regularization through lattice spacing a
- UV cutoff $\sim 1/a$; IR cutoff $\sim 1/aN$



Gauge fields

- Link variables $U_\mu(x) = e^{-aA_\mu(x)} \in \mathrm{SU}(3)$ on links
- Configuration: Snapshot of all $4 \times N_\sigma^3 \times N_\tau$ links



Path integrals

Want expected value of operator O.

- QFT expectation values:

$$\langle O \rangle \sim \int \mathcal{D}U \, e^{iS(U)} \, O(U)$$

- Lattice QCD expectation values:

$$\langle O \rangle \sim \int \mathcal{D}U \, e^{-S(U)} \, O(U)$$

- Achieved through Wick rotation

$$t \to i\tau$$

- Hence our Metric goes from Minkowski to Euclidean

MCMC

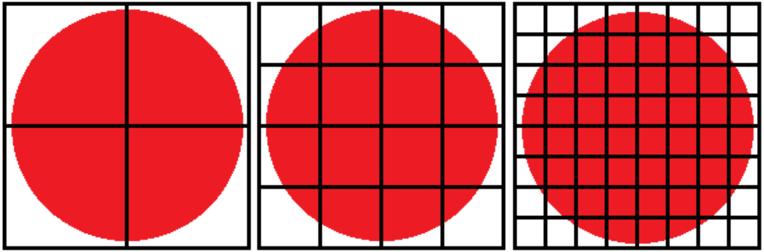
- Markov Chain Monte Carlo basic idea:
  - Each configuration generated depending on last one only
  - Accept new configuration with probability $\min\{1, e^{-\Delta S}\}$
  - Create a time series of measurements O_n of O
- The estimator for $\langle O \rangle$ on the lattice is

$$\bar{O} = \frac{1}{N_{\text{conf}}} \sum_{n=1}^{N_{\text{conf}}} O_n$$

Continuum limit

- Continuum limit: $a \to 0$
- Must also increase the number of sites
- Carry out a fit, usually need 3 or more spacings:

$$O(a) = O^{\text{cont.}} + a^2 O_0$$



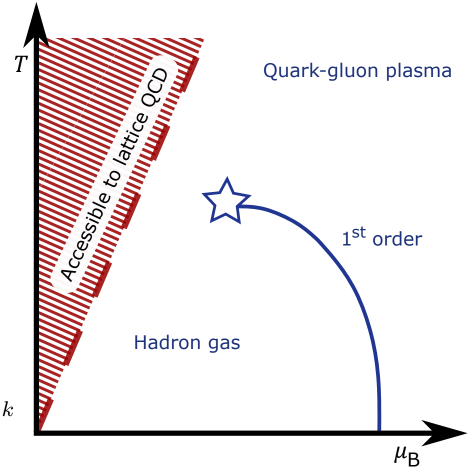
What lattice can do

$$\langle O\rangle \sim \int \mathrm{d}\bar\psi\,\mathrm{d}\psi\, e^{-\bar\psi D\psi} \quad \mathrm{d}U\, e^{-S(U)}\, O(U)$$
$$= \int \det D \quad \mathrm{d}U\, e^{-S(U)}\, O(U)$$

Complication (sign problem):

- $\det D \in \mathbb{R}$ when $\mu = 0$
- But if $\mu \neq 0$, it is complex...
- Can use tricks:

$$\frac{p}{T^4} \sim \sum_{i,j,k=0}^{\infty} \chi_{ijk}^{BQS} \left(\frac{\mu_B}{T}\right)^i \left(\frac{\mu_Q}{T}\right)^j \left(\frac{\mu_S}{T}\right)^k$$



Motivation: Material parameters

Want to learn about composition and properties of strongly interacting systems. One way is through material parameters like isentropic sound speed

$$c_s^2 \equiv \left(\frac{\partial p}{\partial \epsilon}\right)_{s/n_B}$$

Material parameters give physical intuition how strongly interacting matter deforms, expands, etc. Can have more utility: c_s^2 particularly useful, e.g.

- related to fireball expansion rate[a]
- has "soft point" (minimum) near crossover
- check if NS centers contain hadronic d.o.f.[b]

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[a] J. D. Bjorken, Phys. Rev. D, 27, 140–151 (1983).
[b] I. Tews et al., Astrophys. J., 860.2, 149 (2018).

Strategy

GOAL: compute material parameters at $\mu_B > 0$.

But only have *direct* access to $\mu_B = 0$ on the lattice. Commonly played game:

1. Write p/T^4 as **Taylor expansion** in μ_i/T
2. Derive material parameters from p/T^4 using **thermodynamics**
3. **Measure** Taylor coefficients on lattice
4. Compare against **HRG** for $T < T_{\rm pc}$ (crossover temp ~ 156 MeV)
5. Compare against **ideal gas** for $T \gg T_{\rm pc}$

Lattice pressure

For convenience, $\hat{X} \equiv XT^{-k}$ with k s.t. $\hat{X}$ dimensionless (e.g. $\hat{\mu} = \mu/T$)

Dealing with 3 chemical potentials $\hat{\mu}_B$, $\hat{\mu}_Q$, $\hat{\mu}_S$

To make contact with T-$\hat{\mu}_B$ plane, need to eliminate 2 independent variables

Can **impose external constraint**, e.g.

1. $\hat{\mu}_Q = \hat{\mu}_S = 0$
2. $n_S = 0$, $n_Q/n_B = 0.4$ (RHIC-like initial conditions, collide Au nuclei)
3. $n_S = 0$, $n_Q/n_B = 0.5$ (isospin-symmetric; yields $\hat{\mu}_Q = 0$)

and think of **expansions in $\hat{\mu}_B$ only**:

$$\hat{p} \sim \sum_{i,j,k=0}^{\infty} \chi_{ijk}^{BQS}\, \hat{\mu}_B^i\, \hat{\mu}_Q^j\, \hat{\mu}_S^k, \qquad \chi_2^B \equiv \frac{\partial^2 \hat{p}}{\partial \hat{\mu}_B^2} \qquad \to \qquad \hat{p} = \sum_{k \text{ even}} P_k(T) \hat{\mu}_B^k$$

Hadron resonance gas pressure

Non-interacting, quantum, relativistic gas eventually gives (single species)

$$\hat{p}^{\mathrm{HRG}} = \frac{m^2 g}{2\pi^2 T^2} \sum_{k=1}^{\infty} \frac{\eta^{k+1} z^k}{k^2} K_2\left(\frac{mk}{T}\right), \qquad z \equiv e^{\hat{\mu}_B B + \hat{\mu}_Q Q + \hat{\mu}_S S},$$

with K_2 modified Bessel function 2^{nd} kind. HRG:

- Assume such gas where hadrons and resonances only d.o.f.
- Hence valid up to $\sim T_{\mathrm{pc}}$
- Sum over all such states, each with g_i, m_i, etc.
- K_2 exponentially suppressed, so can keep few terms

Ideal gas pressure

Ideal, massless gas of up, down, and strange quarks:

$$\hat{p}^{\rm id} = \frac{19\pi^2}{36} + \frac{1}{2}\left(\hat{\mu}_u^2 + \hat{\mu}_d^2 + \hat{\mu}_s^2\right) + \frac{1}{4\pi^2}\left(\hat{\mu}_u^4 + \hat{\mu}_d^4 + \hat{\mu}_s^4\right),$$

which can be rewritten using

$$\begin{aligned} \mu_u &= \frac{1}{3}\mu_B + \frac{2}{3}\mu_Q \\ \mu_d &= \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q \\ \mu_s &= \frac{1}{3}\mu_B - \frac{1}{3}\mu_Q - \mu_S. \end{aligned}$$

Extracting a material parameter

EXAMPLE: $c_s^2 \sim \left(\frac{\partial p}{\partial \epsilon}\right)_s$ with only one μ.

$$
\begin{aligned}
\mathrm{d}p &= \frac{\partial p}{\partial T}\,\mathrm{d}T + \frac{\partial p}{\partial \mu}\,\mathrm{d}\mu && (1)\\
\mathrm{d}\epsilon &= \frac{\partial \epsilon}{\partial T}\,\mathrm{d}T + \frac{\partial \epsilon}{\partial \mu}\,\mathrm{d}\mu && (2)\\
\mathrm{d}s &= \frac{\partial s}{\partial T}\,\mathrm{d}T + \frac{\partial s}{\partial \mu}\,\mathrm{d}\mu && (3)
\end{aligned}
$$

Think of each $O(T,\mu)$. But also $p(\epsilon, s)$:

$$
\begin{aligned}
\mathrm{d}p &= \frac{\partial p}{\partial \epsilon}\,\mathrm{d}\epsilon + \frac{\partial p}{\partial s}\,\mathrm{d}s\\
&= c_s^2\,\mathrm{d}\epsilon + \frac{\partial p}{\partial s}\,\mathrm{d}s
\end{aligned}
\qquad (4)
$$

Hence use (2) and (3) to eliminate $\mathrm{d}\mu$ and $\mathrm{d}T$ in favor of $\mathrm{d}\epsilon$ and $\mathrm{d}s$ in (1):

$$
c_s^2 = \frac{n^2\frac{\partial s}{\partial T} - 2sn\frac{\partial s}{\partial \mu} + s^2\frac{\partial n}{\partial \mu}}{(\epsilon + p)\left(\frac{\partial s}{\partial T}\frac{\partial n}{\partial \mu} - \frac{\partial s}{\partial \mu}\frac{\partial s}{\partial \mu}\right)},
$$

where

$$
s = \frac{\partial p}{\partial T}, \qquad n = \frac{\partial p}{\partial \mu}
$$

Everything in terms of μ-, T-derivatives of p!

What material parameters we look at

We have μ_B, μ_Q, μ_S. Eventually recast everything in terms of intensive quantities

$$c_s^2 = \left(\frac{\partial p}{\partial \epsilon}\right)_{s/n_B, r, n_S/n_B}$$

$$c_T^2 = \left(\frac{\partial p}{\partial \epsilon}\right)_{T, r, n_S/n_B}$$

$$\kappa_s = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{S,\vec{N}} = \frac{1}{n_B}\left(\frac{\partial n_B}{\partial p}\right)_{s/n_B, r, n_S/n_B}$$

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial p}\right)_{T,\vec{N}} = \frac{1}{n_B}\left(\frac{\partial n_B}{\partial p}\right)_{T, r, n_S/n_B}$$

$$C_V = \frac{T}{V}\left(\frac{\partial \mathcal{S}}{\partial T}\right)_{V,\vec{N}} = T\left(\frac{\partial s}{\partial T}\right)_{n_B, r, n_S/n_B}$$

$$C_p = \frac{T}{V}\left(\frac{\partial \mathcal{S}}{\partial T}\right)_{p,\vec{N}} = n_B T\left(\frac{\partial s/n_B}{\partial T}\right)_{n_B, r, n_S/n_B}$$

$$\alpha = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)_{p,\vec{N}} = -\frac{1}{n_B}\left(\frac{\partial n_B}{\partial T}\right)_{p, r, n_S/n_B}$$

Relations among the parameters

Can be used to cross-check formulae:

$$\frac{C_p}{C_V} = \frac{\kappa_T}{\kappa_s}, \qquad C_p - C_V = \frac{T\alpha^2}{\kappa_T}$$
$$\kappa_T - \kappa_s = \frac{T\alpha^2}{C_p},$$
$$\kappa_s = \frac{1}{c_s^2(\epsilon + p - \mu_Q n_Q - \mu_S n_S)}.$$

Also cross-check against single chemical potential[1].

[1] S. Floerchinger and M. Martinez, Phys. Rev. C, 92.6, 064906 (2015).

An example

For instance, the **isothermal compressibility** comes out to be

$$\kappa_T = \left(\frac{\left(X_{11}^{BS}\right)^2 X_2^Q - 2X_{11}^{BQ} X_{11}^{BS} X_{11}^{QS} + X_2^B \left(X_{11}^{QS}\right)^2 + \left(X_{11}^{BQ}\right)^2 X_2^S - X_2^B X_2^Q X_2^S}{n_B^2 b_{B2}} \right)_{T,r,n_S/n_B}$$

Here $X_2^B(\vec{\mu},T) = \partial_{\mu_B}^2 p$ and $b_{B2}\left(r, X_{ijk}^{BQS}\right)$.

κ_T diverges as $\mu_B \to 0$.

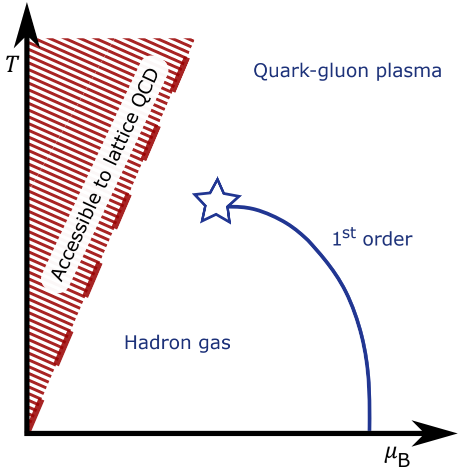
Ideal gas limit (rescaled):

$$n_B^2 \kappa_T^{\rm id} T^{-2} = \frac{2}{27}\left(\frac{\hat{\mu}_B^2}{\pi^2} + 3\right)$$

More uses of material parameters

Discussed already some uses of c_s^2. Also

- Look for critical point
  - c_s^2 hits 0
  - $\kappa_T \sim |T - T^{\text{CEP}}|^{-\gamma}, \quad \gamma \approx 1.23$
  - $C_V \sim |T - T^{\text{CEP}}|^{-\alpha}, \quad \alpha \approx 0.11$
  - location[a,b]: $T^{\text{CEP}} \lesssim 110$, $\mu_B^{\text{CEP}} \gtrsim 420$ MeV
- κ_T relates to n_B fluctuations
- C_V relates to thermal fluctuations



[a] D. A. Clarke et al., PoS, LATTICE2023, 168 (2024).
[b] J. Goswami et al., QM2023, (Jan. 2024).

Some context and lattice setup

Related studies from the past, for instance $\hat{\mu}_B = 0$[2,3] and $\hat{\mu}_B > 0$[4,5].

There you can find c_s^2 and C_V. This study:

- Taylor series up to 6^{th} order; converges well at least for $\hat{\mu}_B \lesssim 2$[6,7]
- Set physical $m_s/m_l = 27$
- Focus[8] on $n_S = 0$ and $r \equiv n_Q/n_B = 0.5$
- First lattice determinations κ_T, c_T^2, C_p, and α

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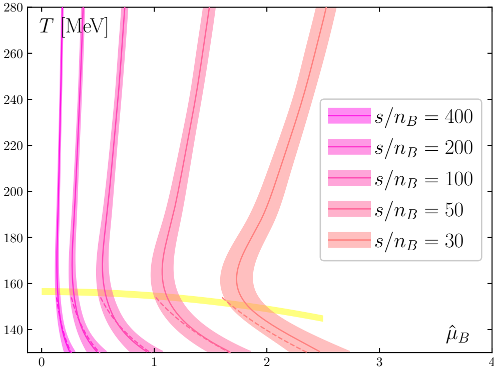
[2] A. Bazavov et al., Phys. Rev. D, 90, 094503 (2014).
[3] S. Borsanyi et al., Phys. Lett. B, 730, 99–104 (2014).
[4] A. Bazavov et al., Phys. Rev. D, 95.5, 054504 (2017).
[5] S. Borsanyi et al., JHEP, 10, 205 (2018).
[6] D. Bollweg et al., Phys. Rev. D, 105.7, 074511 (2022).
[7] Strictly speaking, convergence radius depends on temperature.
[8] Results at $r = 0.4$ and $r = 0.5$ are similar.

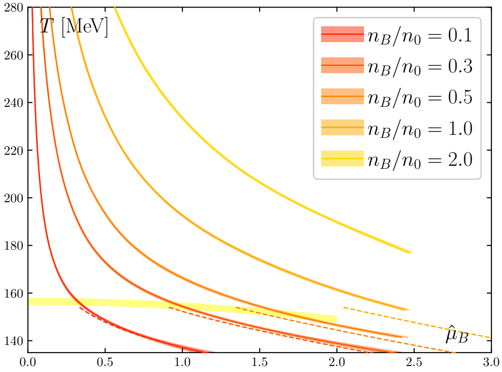
Some context and lattice setup

$$T = \frac{1}{aN_\tau}$$

- $N_f = 2 + 1$ HISQ ensembles
- Set scale with f_K
- Approx. 1.5M, 300k, 22k configs for $N_\tau = 8$, 12, 16, respectively
- Continuum-extrapolated 135 MeV $\leq T \leq$ 175 MeV
- $N_\tau = 8$ outside the range

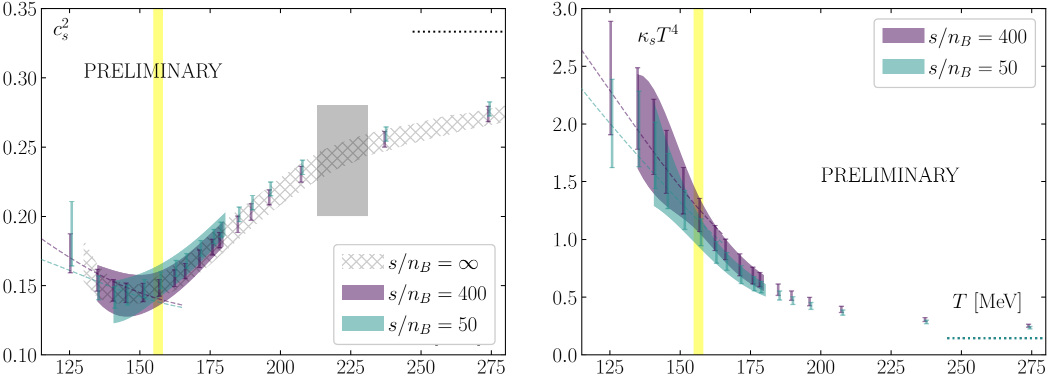
Lines of constant physics $n_S = 0$, $n_Q/n_S = 0.5$





$$n_0 = 0.16\ \text{fm}^{-3}$$

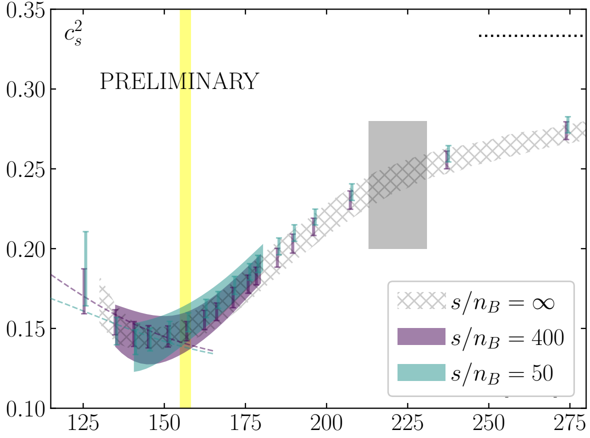
Results: Isentropic observables[9,10]



$$\kappa_s = \frac{1}{c_s^2(\epsilon+p-\mu_Q n_Q-\mu_S n_S)}$$

[9] F. G. Gardim et al., Nature Phys. 16.6, 615–619 (2020).
[10] A. Bazavov et al., Phys. Rev. D, 90, 094503 (2014).

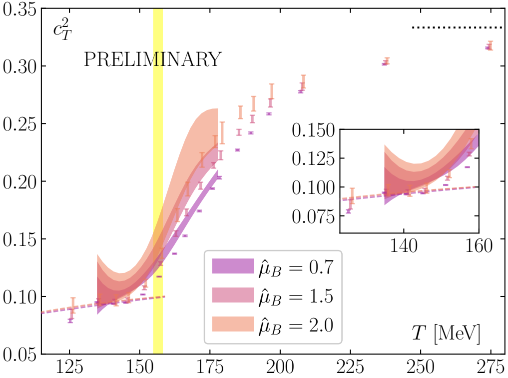
Results: Isentropic observables

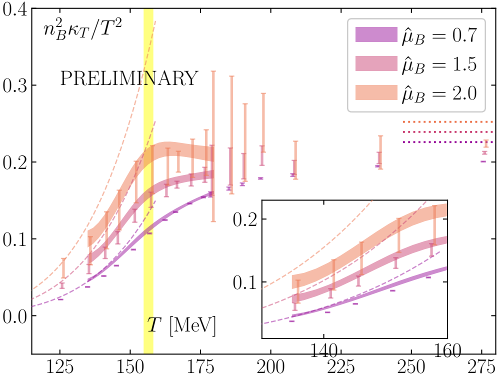


In the surveyed range ($\hat{\mu}_B \lesssim 3$):

- Dip near $T_{\rm pc}$
- Large s/n_B shows good agreement with $\vec{\mu} = 0$
- Minimal dependence on $\vec{\mu}$
- Agreement with Gardim *et al.* extraction
- No violation of conformal limit

Results: Isothermal observables

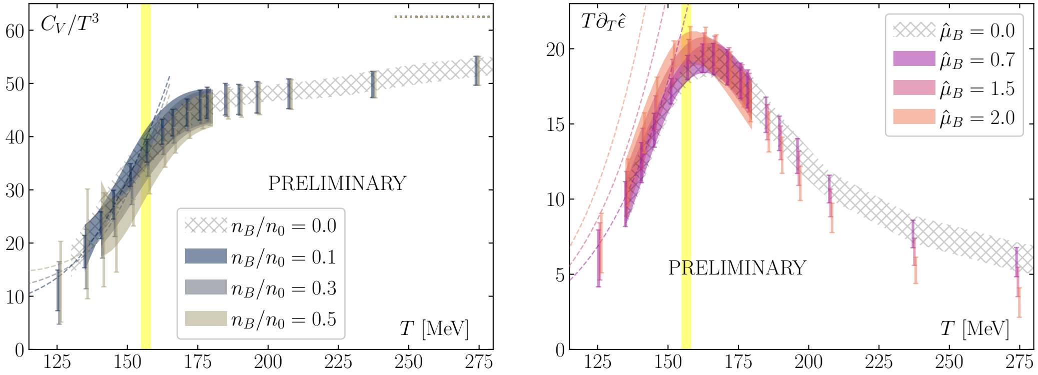




c_T^2 has no dip

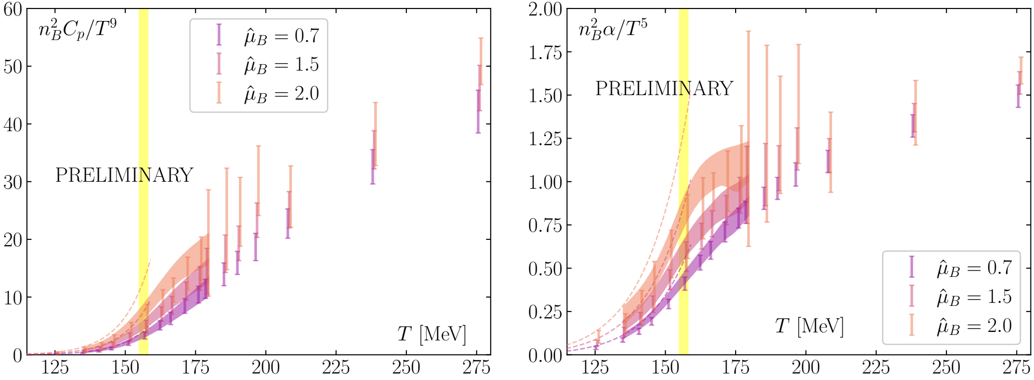
Noticeable $\hat{\mu}_B$ dependence

Results: Isovolumetric specific heat



Mild dependence near $T_{\rm pc}$ on $\hat{\mu}_B$

Results: Isobaric observables



Conclusions

- Derived formulae for material parameters that obey several theoretical cross-checks
- Have lattice data for c_s^2, c_T^2, κ_s, κ_T, C_V, C_V, and α for $\hat{\mu}_B \lesssim 3$
- First lattice determinations of c_T^2, κ_T, C_p, and α
- Speed of sound doesn't exceed conformal limit
- $\hat{\mu}_B$-dependence of many observables somewhat mild
- No critical behavior for $\hat{\mu}_B \lesssim 3$, consistent with $\hat{p}$ convergence radius

Thanks for your attention.

Cutoff effects