Generalizing the isothermal compressibility for QCD

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CRC–TR 211
Strong-interaction matter under extreme conditions

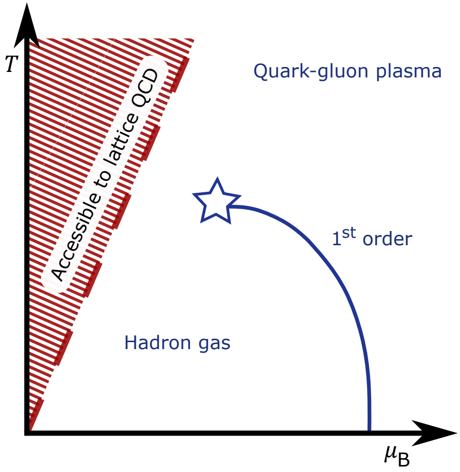
Broadly interested in **phase diagram** of strongly interacting systems.

At high enough temperatures and/or densities, hadrons dissociate to quark-gluon plasma.

Relevant to several systems:

- Early universe
- Neutron stars (NS)
- Heavy ion collisions (HIC)

Look for **critical end point (CEP)**!



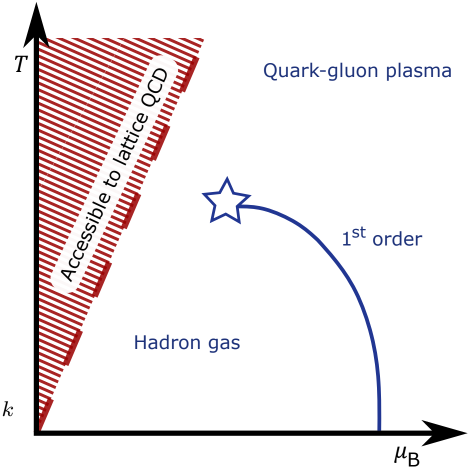
What lattice can do

$$\langle O\rangle \sim \int \mathrm{d}\bar\psi\,\mathrm{d}\psi\, e^{-\bar\psi D\psi} \quad \mathrm{d}U\, e^{-S(U)}\, O(U)$$
$$= \int \det D \quad \mathrm{d}U\, e^{-S(U)}\, O(U)$$

Complication (sign problem):

- $\det D \in \mathbb{R}$ when $\mu = 0$
- But if $\mu \neq 0$, it is complex...
- Can use tricks:

$$\frac{p}{T^4} \sim \sum_{i,j,k=0}^{\infty} \chi_{ijk}^{BQS} \left(\frac{\mu_B}{T}\right)^i \left(\frac{\mu_Q}{T}\right)^j \left(\frac{\mu_S}{T}\right)^k$$



Motivation: Material parameters

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Want to learn about composition and properties of strongly interacting systems. One way is through material parameters like isothermal compressibility

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,N}$$

Material parameters give physical intuition how strongly interacting matter deforms, expands, etc. For κ_T:

- Look for critical point (assuming it exists)

$$\kappa_T \sim |T - T^{\text{CEP}}|^{-\gamma}, \quad \gamma \approx 1.23$$

- location[a,b]: $T^{\text{CEP}} < 100$, $\mu_B^{\text{CEP}} \gtrsim 420$ MeV
- κ_T also relates to n_B fluctuations

[a] D. A. Clarke et al., Phys. Rev. D, 112.9, L091504 (2025).
[b] D. A. Clarke et al., arXiv:2601.04782, (2026).

Strategy

GOAL: compute material parameters at $\mu_B > 0$.

But only have *direct* access to $\mu_B = 0$ on the lattice. Commonly played game:

1. Write p/T^4 as **Taylor expansion** in μ_i/T
2. Derive material parameters from p/T^4 using **thermodynamics**
3. **Measure** Taylor coefficients on lattice
4. Compare against **HRG** for $T < T_{\rm pc}$ (crossover temp ~ 156 MeV)

Lattice pressure

For convenience, $\hat{X} \equiv XT^{-k}$ with k s.t. $\hat{X}$ dimensionless (e.g. $\hat{\mu} = \mu/T$)

Dealing with 3 chemical potentials $\hat{\mu}_B$, $\hat{\mu}_Q$, $\hat{\mu}_S$

To make contact with T-$\hat{\mu}_B$ plane, need to eliminate 2 independent variables

Can **impose external constraint**, e.g.

1. $\hat{\mu}_Q = \hat{\mu}_S = 0$
2. $n_S = 0$, $n_Q/n_B = 0.4$ (RHIC-like initial conditions, collide Au nuclei)
3. $n_S = 0$, $n_Q/n_B = 0.5$ (isospin-symmetric; yields $\hat{\mu}_Q = 0$)

and think of **expansions in $\hat{\mu}_B$ only**:

$$\hat{p} \sim \sum_{i,j,k=0}^{\infty} \chi_{ijk}^{BQS}\, \hat{\mu}_B^i\, \hat{\mu}_Q^j\, \hat{\mu}_S^k, \qquad \chi_2^B \equiv \frac{\partial^2 \hat{p}}{\partial \hat{\mu}_B^2}\bigg|_{\mu=0} \qquad \rightarrow \qquad \hat{p} = \sum_{k \text{ even}} P_k(T)\hat{\mu}_B^k$$

Hadron resonance gas pressure

Non-interacting, quantum, relativistic gas eventually gives (single species)

$$\hat{p}^{\mathrm{HRG}} \sim m^2 g \sum_{k=1}^{\infty} \frac{\eta^{k+1} z^k}{k^2} K_2\left(\frac{mk}{T}\right), \qquad z \equiv e^{\hat{\mu}_B B + \hat{\mu}_Q Q + \hat{\mu}_S S},$$

with K_2 modified Bessel function 2^{nd} kind. HRG:

- Assume such gas where hadrons and resonances only d.o.f.
- Hence valid up to $\sim T_{\mathrm{pc}}$
- Sum over all such states, each with g_i, m_i, etc.
- K_2 exponentially suppressed, so can keep few terms

A problem for κ_T in field theories

With one chemical potential:

$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,N} = \frac{1}{n^2}\frac{\partial^2 P}{\partial \mu^2}, \qquad n = N/V$$

With B, Q, and S, $n_S = 0$, isothermal compressibility comes out to be

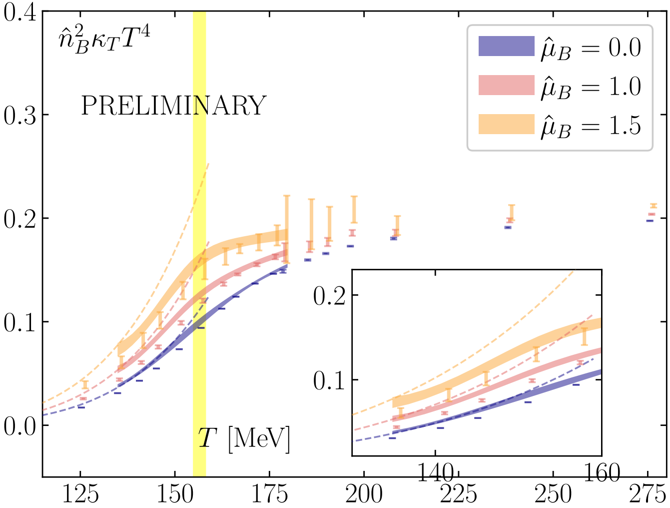
$$\kappa_T = \frac{F\left(X_{ijk}^{BQS}\right)}{n_B^2\, G\left(n_Q/n_B, X_{ijk}^{BQS}\right)},$$

where e.g. $X_2^B \equiv \partial^2_{\hat{\mu}_B} p(\mu_B)$ depends on μ_B. Since $n_B \propto \mu_B$

κ_T diverges[1] as $\mu_B \to 0$

[1] Also a problem for material parameters like C_p and α.

A problem for κ_T in field theories



A solution

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$$\kappa_T = -\frac{1}{V}\left(\frac{\partial V}{\partial P}\right)_{T,N} = \frac{1}{n^2}\frac{\partial^2 P}{\partial \mu^2}$$

Fix instead f s.t. $f \neq 0$ at $\mu = 0$:

$$\kappa_{T,f} T^4 = -\frac{T^4}{V}\left(\frac{\partial V}{\partial P}\right)_{T,f,n_Q/n_B,n_S/n_B}$$

$$\kappa_{T,f} T^4 = -\frac{1}{f\hat{n}_B}\left(\frac{\partial f}{\partial \hat{\mu}_B}k_B + \frac{\partial f}{\partial \hat{\mu}_Q}k_Q + \frac{\partial f}{\partial \hat{\mu}_S}k_S\right)$$

at $n_S = 0$ where $k_i\left(n_Q/n_B, X^{BQS}_{ijk}\right)$.

How to pick f?

$$\kappa_{T,f} T^4 = -\frac{1}{f \hat{n}_B}\left(\frac{\partial f}{\partial \hat{\mu}_B} k_B + \frac{\partial f}{\partial \hat{\mu}_Q} k_Q + \frac{\partial f}{\partial \hat{\mu}_S} k_S\right)$$

Take e.g. $f = V X_2^Q$. Advantages:

- **Meaningful** across $T_{\rm pc}$
- Leads at $\mathcal{O}(\mu^2)$, so one derivative leads at $\mathcal{O}(\mu) \Rightarrow$ **finite**
- A good proxy for $N + \bar{N}$ in hadronic phase[2] $\Rightarrow$ **probes similar physics**

Comes out to be

$$\kappa_{T,\sigma_Q^2} T^4 = \frac{\hat{X}_{12}^{BQ}}{\hat{n}_B \hat{X}_2^Q}\left(1 - \frac{\hat{X}_{21}^{QS}\hat{X}_{11}^{BS}}{\hat{X}_{12}^{BQ}\hat{X}_2^S}\right)$$

[2] P. Braun-Munzinger et al., Phys. Lett. B, 747, 292–298 (2015).

Some context and lattice setup

Related studies from the past, for instance $\hat{\mu}_B = 0$[3,4] and $\hat{\mu}_B > 0$[5,6].

These studies compute c_s^2 and C_V. This study:

- Taylor series up to 6^{th} order; converges well at least for $\hat{\mu}_B \lesssim 2$[7,8]
- $N_f = 2 + 1$ with physical $m_s/m_l = 27$
- Focus[9] on $n_S = 0$ and $n_Q/n_B = 0.5$
- First lattice determination of κ_T

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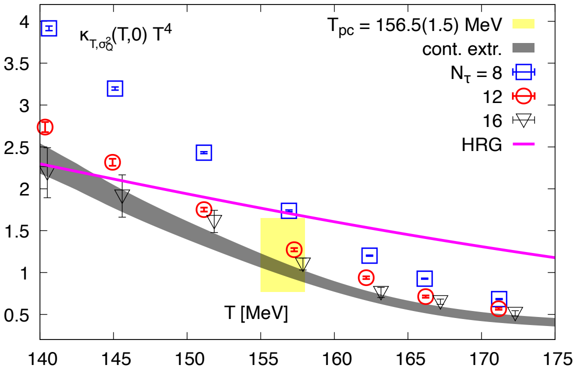
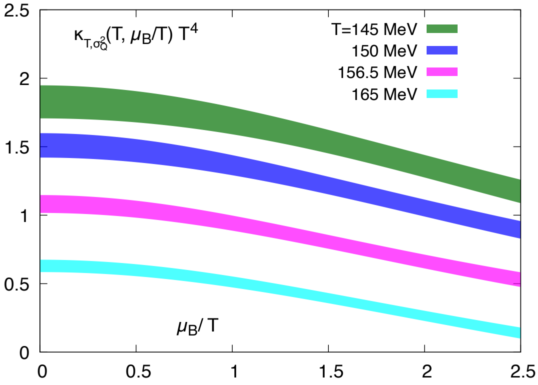
[3] A. Bazavov et al., Phys. Rev. D, 90, 094503 (2014).
[4] S. Borsanyi et al., Phys. Lett. B, 730, 99–104 (2014).
[5] A. Bazavov et al., Phys. Rev. D, 95.5, 054504 (2017).
[6] S. Borsanyi et al., JHEP, 10, 205 (2018).
[7] D. Bollweg et al., Phys. Rev. D, 105.7, 074511 (2022).
[8] Strictly speaking, convergence radius depends on temperature.
[9] Results at $n_Q/n_B = 0.4$ and $n_Q/n_B = 0.5$ are similar.





Decreasing function of μ_B and T

Difference with HRG may be traceable back to Δ^{++} resonance

[10] D. A. Clarke et al., arXiv:2506.22816, (2025).

Contact with experiment

In HRG, κ_T can be expressed as[11]

$$\kappa_{T,\vec{N}}^{-1} = -V\left(\frac{\partial P}{\partial V}\right)_{T,\vec{N}} = T^4\left(\sum_{i\in\text{charged}}\frac{\hat{n}_i}{\omega_i} + \sum_{i\in\text{neutral}}\frac{\hat{n}_i}{\omega_i}\right), \qquad \omega_i \equiv \sigma_i^2/N_i$$

Variances ω_i similar, so to a good approximation,

$$\kappa_{T,\vec{N}}^{-1} \approx T^4\left(\frac{\hat{n}_\text{ch}}{\omega_\text{ch}} + \frac{\hat{n}_0}{\omega_0}\right) \approx T\frac{n_\text{tot}}{\omega_\text{tot}},$$

where

$$\hat{n}_X = \sum_{i\in X}\hat{n}_i \quad , \quad \omega_X = \frac{\sum_{i\in X}\sigma_i^2}{\sum_{i\in X}N_i} \quad , \quad X = \text{ch}, 0, \text{tot}$$

[11] M. Mukherjee et al., Phys. Lett. B, 784, 1–5 (2018).

Contact with experiment

$$\kappa_{T,\vec{N}}^{-1} \approx T^4 \left(\frac{\hat{n}_{\rm ch}}{\omega_{\rm ch}} + \frac{\hat{n}_0}{\omega_0} \right) \approx T \frac{n_{\rm tot}}{\omega_{\rm tot}},$$

- Neutral term often neglected in calculations from HIC
- Plugging in HRG pressure at $T_{\rm pc}$ yields $p\kappa_{T,\vec{N}} \approx 1.03$
- For an ideal gas, $p\kappa_{T,\vec{N}} = 1$

$\kappa_{T,f}$ along pseudocritical line

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$T_{\rm pc}$ is a decreasing function of μ_B:

$$T_{\rm pc}(\hat{\mu}_B) = T_{\rm pc,0}\left(1 - \kappa_2\hat{\mu}_B^2 + \mathcal{O}(\hat{\mu}_B^4)\right)$$

From lattice[12], $T_{\rm pc,0} = 156.5(1.5)$ MeV and $\kappa_2 = 0.012(4)$. Plug this expansion into expression for κ_{T,σ_Q^2}. Find

$$p\kappa_{T,\sigma_Q^2}\Big|_{T_{pc}(\hat{\mu}_B)} = 0.80(12) + 0.06(1)\hat{\mu}_B^2 + \mathcal{O}(\hat{\mu}_B^4)$$

Both definitions suggest similarity to ideal gas

[12] A. Bazavov et al., Phys. Lett. B, 795, 15–21 (2019).

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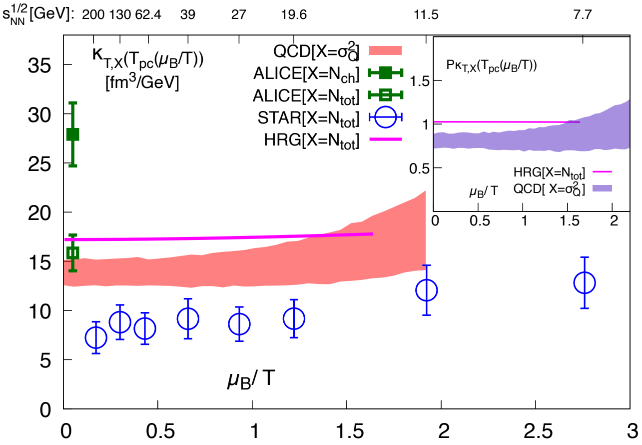

- $p\kappa_{T,\sigma_Q^2}$ and $p\kappa_T$ from HRG are close within uncertainty
- Agreement with ALICE using both charged and neutral hadrons
- Compressibility computed from STAR using both charged and neutral hadrons slowly increases with μ_B/T
- Increase seems driven by N_{tot}; no indication of characteristic divergence/CEP

[13] D. A. Clarke et al., arXiv:2506.22816, (2025).

Conclusions

- Introduced $\kappa_{T,f}$ finite at $\mu = 0$ in field theories!
- Consistent with ALICE
- κ_T from STAR shows no critical behavior
- κ_{T,σ_Q^2} on pseudocritical line stays close to that of an ideal gas

Thanks for your attention.

Deviations from HRG

$$\kappa_{T,\sigma_Q^2} T^4 \propto \frac{\hat{X}_{12}^{BQ}}{\hat{n}_B \hat{X}_2^Q} = \frac{\chi_{22}^{BQ}}{\chi_2^B \chi_2^Q} \left(\frac{1 - \frac{\chi_{121}^{BQS}\chi_{11}^{BS}}{\chi_{22}^{BQ}\chi_2^S} + \mathcal{O}(\vec{\hat{\mu}}^2)}{1 - \frac{(\chi_{11}^{BS})^2}{\chi_2^B\chi_2^S} + \mathcal{O}(\vec{\hat{\mu}}^2)} \right)$$